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Hermitian Forms and Locally Toroidal Regular Polytopes

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1. INTRODUCTION

In the classical theory of regular polytopes the structure of a finite or infinite polytope (honeycomb) is governed by a real quadratic form. This form determines a geometry into which the polytope is embedded in such a way that all its symmetries are realized by isometries. The symmetry group is the Coxeter group associated with the quadratic form (cf. Coxeter [8]). These facts have important consequences for other fields in mathematics; see, for example, Tits [29, 30].

The purpose of this paper is to show that much of the correspondence between polytopes and forms remains true for the theory of abstract regular polytopes, that is, regular incidence-polytopes. The concept of regular incidence-polytopes provides a suitable setting for combinatorial structures resembling the classical regular polytopes (cf. Danzer–Schulte [13]); for related notions see also McMullen [17], Grünbaum [16], Dress [14], Buekenhout [2], and Tits [29].

In [16] Grünbaum suggested studying abstract regular polytopes whose faces and vertex-figures are not necessarily of spherical type. He posed the problem of classifying all finite universal (or, in his notation, naturally generated) abstract regular polytopes $\{\mathscr{P}_1, \mathscr{P}_2\}$ in 4 dimensions whose 3-faces $\mathscr{P}_1$ and vertex-figures $\mathscr{P}_2$ are spherical and/or toroidal; see also Coxeter–Shephard [11], Weiss [31, 32], and [24, 25]. In [20, 21] this problem was attacked by twisting operations on Coxeter groups and unitary reflexion groups, leading to the explicit recognition of the universal $\{\mathscr{P}_1, \mathscr{P}_2\}$ for many choices of $\mathscr{P}_1$ and $\mathscr{P}_2$.

In this paper we completely classify all the finite universal polytopes of

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type $\{6, 3, p\}$ with $p \leqslant 6$. This is done by associating with the polytopes $\{\mathscr{P}_1, \mathscr{P}_2\}$ a complex Hermitian form h. This form is positive definite if and only if the universal $\{\mathscr{P}_1, \mathscr{P}_2\}$ is finite. If $\{\mathscr{P}_1, \mathscr{P}_2\}$ is finite, its group contains as a subgroup of small index a unitary reflexion group generated by reflexions of period 2; this group is among the groups classified in Shephard–Todd [28] or Coxeter [6]. The group of $\{\mathscr{P}_1, \mathscr{P}_2\}$ itself is a group of linear and semi-linear unitary isometries. In the general case $\{\mathscr{P}_1, \mathscr{P}_2\}$ has a quotient $\mathscr{P}$ with facets $\mathscr{P}_1$ and vertex-figures $\mathscr{P}_2$, whose group is a group of linear and semi-linear isometries with respect to h. We conjecture that $\mathscr{P} = \{\mathscr{P}_1, \mathscr{P}_2\}$, but we have not been able to prove this for indefinite h. In any case, $\mathscr{P}$ is finite if and only if $\{\mathscr{P}_1, \mathscr{P}_2\}$ is finite.

In Section 2 we recall some facts from the theory of regular incidence-polytopes. Section 3 deals with Hermitian forms and unitary reflexion groups. In Section 4 we discuss the type $\{6, 3, 3\}$ which is the most interesting case. The types $\{6, 3, 4\}$, $\{6, 3, 5\}$, and $\{6, 3, 6\}$ are studied in Section 5. In Section 6 we show how our methods can be used to prove partial results for the type $\{3, 6, 3\}$. In Section 7 we discuss some general aspects of geometrical realizations of the polytopes. Finally, in Section 8 we deal with an exceptional case which we exclude from the discussion in the earlier sections.

2. Regular Incidence-Polytopes

Following [13] a *d-incidence-polytope* $\mathscr{P}$ is a partially ordered set, with a strictly monotone rank function $\dim(\cdot)$ with range $\{-1, 0, \ldots, d\}$. The elements of rank i are called the *i-faces* of $\mathscr{P}$. The *flags* (maximal totally ordered subsets) of $\mathscr{P}$ all contain exactly $d+2$ faces, including the unique (least) (-1)-face F_{-1} and the unique (greatest) d-face F_d of $\mathscr{P}$. Further defining properties for $\mathscr{P}$ are the strong flag-connectedness as well as the property that for any $(i-1)$-face F and any $(i+1)$-face G with $F < G$ there are exactly two i-faces H of $\mathscr{P}$ with $F < H < G$.

If F and G are faces with $F < G$, we call $G/F := \{H \mid F \leqslant H \leqslant G\}$ a *section* of $\mathscr{P}$. There is little possibility of confusion if we identify a face F with the section F/F_{-1}. The faces of dimension 0, 1, and $d-1$ are also called the *vertices*, *edges*, and *facets* of $\mathscr{P}$, respectively. If F is a face, then F_d/F is said to be the *co-face* of F, or the *vertex-figure* of F if F is a vertex.

A d-incidence-polytope $\mathscr{P}$ is *regular* if its automorphism group $A(\mathscr{P})$ is transitive on the flags. For a regular $\mathscr{P}$ its group $A(\mathscr{P})$ is generated by involutions $\rho_0, \ldots, \rho_{d-1}$, where ρ_i is the unique automorphism which keeps fixed all but the i-face F_i of some base flag of $\mathscr{P}$ $(i = 0, \ldots, d-1)$. These *distinguished* generators satisfy relations

$$(\rho_i \rho_j)^{p_{ij}} = 1 \qquad (i, j = 0, \ldots, d-1); \tag{1}$$

here $p_{ii}=1$, $p_{ji}=p_{ij}=: p_{i+1}$ if $j=i+1$, and $p_{ij}=2$ otherwise; the p_i's are given by the *type* $\{p_1, ..., p_{d-1}\}$ of $\mathscr{P}$. The group $A(\mathscr{P})$ has the *intersection property* (2) with respect to the ρ_i's:

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \qquad \text{for} \quad I, J \subset \{0, ..., d-1\}. \tag{2}$$

A group A is called a *C-group* (C here stands for Coxeter) if it is generated by involutions $\rho_0, ..., \rho_{d-1}$ such that (1) and (2) hold. It can be shown that the C-groups are precisely the groups of regular incidence-polytopes (cf. [24]).

Given regular d-incidence-polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$ such that the vertex-figures of $\mathscr{P}_1$ are isomorphic to the facets of $\mathscr{P}_2$, we denote by $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ the class of regular $(d+1)$-incidence-polytopes $\mathscr{P}$ with facets isomorphic to $\mathscr{P}_1$ and vertex-figures isomorphic to $\mathscr{P}_2$. If $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ is non-empty, then any such $\mathscr{P}$ is obtained from a *universal* member of $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ by identifications; we denote the universal incidence-polytope by $\{\mathscr{P}_1, \mathscr{P}_2\}$ (cf. [26]).

In most circumstances $\mathscr{P}_1$ and $\mathscr{P}_2$ will be (reflexible) regular maps on surfaces (cf. [10]). The regular maps on the 2-sphere are given by the Platonic solids. On the torus there are the infinite series $\{4, 4\}_{b,c}$, $\{6, 3\}_{b,c}$, and $\{3, 6\}_{b,c}$, with $b=c \geqslant 1$ or $c=0$, $b \geqslant 1$. However, the maps $\{4, 4\}_{1,0}$, $\{4, 4\}_{1,1}$, $\{6, 3\}_{1,0}$, and $\{3, 6\}_{1,0}$ are not 3-incidence-polytopes in the sense defined above, so that they will be excluded from further discussion. Also, in our considerations the maps $\{3, 6\}_{1,1}$ and $\{6, 3\}_{1,1}$ will play an exceptional role. In Sections 3 to 7, whenever a toroidal map $\{3, 6\}_{b,c}$ occurs, we are assuming that $(b, c) \neq (1, 1)$. The case $(b, c)=(1, 1)$ will be discussed separately in Section 8.

For the maps $\{6, 3\}_{b,c}$ the extra relations for the generators ρ_i are

$$\begin{cases} (\rho_2(\rho_1\rho_0)^2)^{2c}=1 & \text{if} \quad b=c \geqslant 1; \\ (\rho_0\rho_1\rho_2)^{2b}=1 & \text{if} \quad c=0,\, b \geqslant 1. \end{cases} \tag{3}$$

A regular d-incidence-polytope $\mathscr{P}$ is called *locally of genus* (*at most*) g if all its 3-dimensional sections are regular maps on orientable surfaces of genus at most g and at least one section is a map of genus g. We call $\mathscr{P}$ *locally toroidal* if it is locally of genus 1. Note that a locally toroidal 4-dimensional $\mathscr{P}$ is necessarily of type $\{3, 4, 4\}$, $\{4, 4, 3\}$, $\{4, 4, 4\}$, $\{3, 6, 3\}$, $\{6, 3, p\}$, or $\{p, 3, 6\}$, with $p=3, 4, 5$, or 6 (provided all entries in the symbol are greater than 2).

Several times we make use of the following simple lemmas.

LEMMA 1. *Let $\mathscr{M}$ be a regular map of type $\{p, 3\}$, $p \geqslant 2$. Then the universal $\mathscr{L} := \{\mathscr{M}, \{3, 6\}_{2,0}\}$ exists if and only if the universal $\mathscr{P} := \{\mathscr{M}, \{3, 3\}\}$ exists; if $\mathscr{L}$ exists, then $A(\mathscr{L}) = A(\mathscr{P}) \times C_2$.*

Proof. First observe that the group of $\{3, 6\}_{2,0}$ is $S_4 \times C_2$ (cf. [10]). Now, let $\mathscr{P}'$ be any member of $\langle \mathscr{M}, \{3, 3\} \rangle$, with group $A(\mathscr{P}') = \langle \rho_0, ..., \rho_3 \rangle$; let $C_2 = \langle \alpha \rangle$ (say). Then $A(\mathscr{P}') \times C_2$ is the group of a regular 4-incidence-polytope $\mathscr{L}'$ in $\langle \mathscr{M}, \{3, 6\}_{2,0} \rangle$, with distinguished generators

$$\varphi_0 := (\rho_0, 1), \qquad \varphi_1 := (\rho_1, 1), \qquad \varphi_2 := (\rho_2, 1), \qquad \varphi_3 := (\rho_3, \alpha).$$

Vice versa, if $\mathscr{L}'$ is any member in $\langle \mathscr{M}, \{3, 6\}_{2,0} \rangle$, with group $A(\mathscr{L}') = \langle \varphi_0, ..., \varphi_3 \rangle$ (say), then $\alpha := (\varphi_2 \varphi_3)^3$ is in the center of $A(\mathscr{L}')$, and $A(\mathscr{L}') = \langle \varphi_0, \varphi_1, \varphi_2, \varphi_3 \alpha \rangle =: A$ or $A(\mathscr{L}') = \langle \alpha \rangle \times A$; but A and its generators define a member $\mathscr{P}'$ of $\langle \mathscr{M}, \{3, 3\} \rangle$ (see Lemma 2). If $\mathscr{L}'$ is universal, then $A(\mathscr{L}') \neq A$, by the first part of the proof applied to $\mathscr{P}'$. Now Lemma 1 follows.

LEMMA 2. *Let A be generated by involutions $\rho_0, ..., \rho_{d-1}$ $(d \geq 4)$ with property* (1) *such that $\langle \rho_0, ..., \rho_{d-2} \rangle$ and $\langle \rho_1, ..., \rho_{d-1} \rangle$ are C-groups. Assume $\langle \rho_1, ..., \rho_{d-1} \rangle$ defines the $(d-1)$-simplex $\{3^{d-2}\}$. Then A is a C-group if and only if $\rho_{d-1} \notin \langle \rho_0, ..., \rho_{d-2} \rangle$.*

Proof. Since $\langle \rho_0, ..., \rho_{d-2} \rangle$ and $\langle \rho_1, ..., \rho_{d-1} \rangle$ are C-groups, (2) is equivalent to

$$\langle \rho_0, ..., \rho_{d-2} \rangle \cap \langle \rho_1, ..., \rho_{d-1} \rangle = \langle \rho_1, ..., \rho_{d-2} \rangle.$$

But the left side contains $\langle \rho_1, ..., \rho_{d-2} \rangle$ $(= S_{d-1})$, and this subgroup is maximal in $\langle \rho_1, ..., \rho_{d-1} \rangle$ $(= S_d)$. Hence the equality holds if and only if $\rho_{d-1} \notin \langle \rho_0, ..., \rho_{d-2} \rangle$.

In many situations the incidence-polytopes are constructed by so-called *twisting operations* on groups W which are generated by involutions σ_i and admit certain involutory outer automorphisms τ permuting these generators (cf. [20, 21]). Then, in suitable cases we can augment W by the group B generated by the τ's and recognize this semi-direct product A of W by B as the group of a regular incidence-polytope. Such a twisting operation κ on W will usually be denoted by

$$\kappa: (\sigma_1, ..., \sigma_k; \tau_1, ..., \tau_l) \rightarrow (\rho_0, ..., \rho_{d-1});$$

here $\sigma_1, ..., \sigma_k$ are the generators of W, $\tau_1, ..., \tau_l$ the generating outer automorphisms in B, and $\rho_0, ..., \rho_{d-1}$ the distinguished generators of A. In most cases the τ's are induced by diagram symmetries; in the diagrams these will usually be indicated by arrows.

In discussing *geometrical realizations* of regular incidence-polytopes $\mathscr{P}$ we start from a suitable faithful linear or affine representation of $A(\mathscr{P}) = \langle \rho_0, ..., \rho_{d-1} \rangle$ and apply Wythoff's construction (cf. [8, 18]). The representation will be real or complex, and the image G of $A(\mathscr{P})$ will usually be a group of linear and semi-linear isometries with respect to some Hermitian form.

Let $G = \langle R_0, ..., R_{d-1} \rangle$, with R_i the affine reflexion in the underlying space E which corresponds to ρ_i. To describe the construction we shall find it convenient to identify a reflexion R with its mirror $\{x \in E \mid R(x) = x\}$ (which is not necessarily a hyperplane). Now, pick a point F_0 (say) in the *Wythoff space* $R_1 \cap \cdots \cap R_{d-1}$ of G and define

$$F_j := \{ g(F_0) \mid g \in \langle R_0, ..., R_{j-1} \rangle \}$$

for $j = -1, 0, ..., d$, with appropriate interpretations for $j = -1, 0$. This gives the base flag $\Phi := \{F_{-1}, F_0, ..., F_d\}$ of the realization for $\mathscr{P}$, whose remaining flags are the $g(\Phi)$ with $g \in G$. The group G becomes the "symmetry group" of the realization. Note that the realization of $\mathscr{P}$ degenerates completely if also $F_0 \in R_0$; we usually exclude this case.

If our representation is complex, we can obtain a real representation of twice the dimension in the standard way. However, we often find it more convenient to work with complex rather than real coordinates. Note also that our notion of realization deviates from that used in [18].

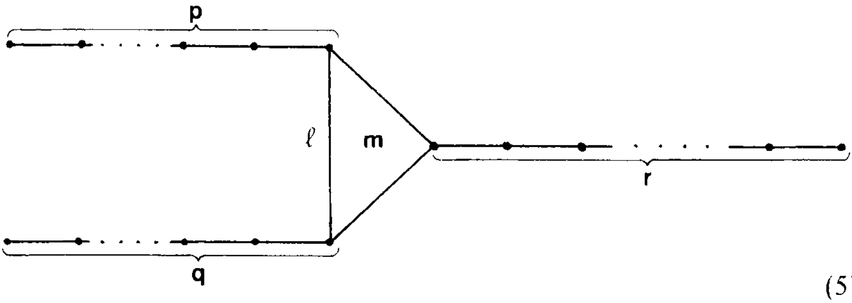
3. Hermitian Forms and Unitary Reflexion Groups

The finite unitary complex groups generated by hyperplane reflexions have been completely classified in Shephard–Todd [28]; see also [3, 6, 27]. Here we shall consider only groups which are generated by reflexions of period 2. In discussing these groups we follow the approach of Coxeter [6, 7].

The finite unitary groups W generated by n involutory reflexions σ_i can conveniently be represented by diagrams. As for Coxeter groups (cf. [1, 10]), each σ_i is symbolized by a node (labelled i), and each pair (σ_i, σ_j) of distinct non-commuting generators is joined by a branch marked with the period m_{ij} of their product; this mark is omitted if $m_{ij} = 3$. For any irreducible group, the σ_i's may be so chosen that the underlying graph is a tree (in which case W is a finite Coxeter group) or contains just one triangular circuit with at least two unmarked branches. In the latter case the graphical symbol for W is completed by writing a number m inside the triangle; see (5) for an example. If i, j, and k (say) are the nodes of a triangular circuit, then a set of defining relations for W is obtained by adding to the *standard relations* (given by the underlying Coxeter diagram) the one extra relation

$$(\sigma_i \sigma_j \sigma_k \sigma_j)^m = 1, \tag{4}$$

or any of the five equivalent relations obtained by relabelling the nodes. All finite non-real unitary groups generated by n involutory reflexions in unitary n-space U_n are groups $[p\,q\,r^l]^m$ in unitary $(p+q+r)$-space, with diagram



(5)

But the actual values for p, q, r, l, and m are restricted to a few choices (cf. [6]); see also our Table I below.

However, for the other parameter values the diagram (5) still makes sense if we interpret it as representing the group W abstractly defined by the standard relations (5) and the extra relation corresponding to (4); this abstract group will also be denoted by $[p\,q\,r^l]^m$. In fact, we shall even consider more general diagrams $\mathscr{D}$ (and corresponding abstract groups $W = W(\mathscr{D})$) consisting of a labelled simplicial 2-complex whose edge graph is a Coxeter diagram, all of whose triangles are marked by m indicating a relation of type (4); see (26) for an example. More generally, if I is the node set of $\mathscr{D}$ and U is a group generated by involutions R_i $(i \in I)$, we use the notation "U belongs to $\mathscr{D}$" if $\sigma_i \to R_i$ defines a homomorphism of $W(\mathscr{D})$ onto U; that is, the R_i's satisfy all defining relations for $W(\mathscr{D})$ in terms of

TABLE I

Non-real Unitary Groups $[1\ 1\ 1^l]^m$

l	m	$c(l, m)$	$2\cos(\pi/l) = \lvert c(l, m)\rvert$	$2\cos(\pi/m) = \lvert 1 + c(l, m)\rvert$	Group order	
3	m	$e^{2\pi i/m} =: c_m$	1	$2\cos(\pi/m)$	$6m^2$	Isomorphic groups if $l = m$
l	3	$-1 - e^{-2\pi i/l}$	$2\cos(\pi/l)$	1	$6l^2$	
4	4	$(-1 + i\cdot\sqrt{7})/2$	$\sqrt{2}$	$\sqrt{2}$	336	
4	5	$-1 - \tau\bar{\omega}$	2	τ	2160	Isomorphic groups
5	4	$\tau\omega$	τ	2	2160	

Note. Here, $\tau = (1+\sqrt{5})/2$ and $\omega = (-1+i\cdot\sqrt{3})/2$, so that $\tau^2 - \tau - 1 = 0$ and $\omega^2 + \omega + 1 = 0$.

the σ_i's, but possibly other independent relations too. As a typical application, $U = \langle R_i \mid i \in I \rangle$ is a reflexion group associated with some Hermitian form as described below. We reserve the notation W and σ_i for the abstract groups, and U and R_i for the geometrical groups. In the case of a finite unitary group we use both notations equivalently.

In complex space $\mathbb{C}^n$, any finite group of linear transformations leaves invariant a positive definite Hermitian form

$$h(x) = \sum_{j,k=1}^{n} a_{jk} x_j \bar{x}_k \qquad (a_{jk} = \bar{a}_{kj}), \tag{6}$$

with $x = (x_1, ..., x_n)$ the coordinate vector with respect to some basis of $\mathbb{C}^n$. For a group U generated by n involutory reflexions $R_1, ..., R_n$ in the (standard) unitary space U_n, we can choose a basis in such a way that the kth reflexion R_k leaves invariant all the coordinates x_j except for x_k, and transforms x_k into

$$x_k - \frac{2}{a_{kk}} \cdot \sum_{j=1}^{n} a_{jk} x_j \tag{7}$$

$$\left(= -x_k + \sum_{j \neq k} \bar{c}_{kj} x_j \text{ in the notation of (11)} \right);$$

necessarily, $a_{kk} \neq 0$ for all k (cf. [6, p. 245]). Thus R_k is a reflexion in the hyperplane

$$\sum_{j=1}^{n} a_{jk} x_j = 0. \tag{8}$$

Occasionally we shall think of the x_k's as contravariant coordinates. Then, introducing the new covariant coordinates

$$x^j = \sum_{k=1}^{n} a_{jk} \bar{x}_k$$

we can write (6) in the form

$$h(x) = \sum_{j=1}^{n} x_j x^j. \tag{9}$$

(Note that we have interchanged the upper and lower indices of [6] for contravariant and covariant coordinates.) Then, by (8), R_k is the reflexion in the covariant hyperplane $x^k = 0$ and changes the covariant coordinate vector $(x^1, ..., x^n)$ to $(y^1, ..., y^n)$ with

$$y^j = x^j - \frac{2}{a_{kk}} \cdot a_{jk} x^k \qquad (j = 1, ..., n). \tag{10}$$

It follows from (10) that R_k leaves invariant the adjoint form

$$\sum_{j,k=1}^{n} a^{jk} x^j \bar{x}^k$$

of (6), with the matrix $(a^{jk})_{j,k}$ the inverse of $(\bar{a}_{ij})_{i,j}$.

If h is any Hermitian form as in (6), not necessarily positive definite but with $a_{kk} \neq 0$ for all k, then (7) still defines reflexions $R_1, ..., R_n$ with respect to h, so that $U = \langle R_1, ..., R_n \rangle$ is a group of isometries with respect to h. Clearly, U is reducible if h is degenerate. We often make use of the following simple lemma.

LEMMA 3. *If h is a non-degenerate indefinite Hermitian form on $\mathbb{C}^n$ and U a subgroup of the isometry group of h acting irreducibly on $\mathbb{C}^n$, then U is infinite.*

Proof. In fact, if U is finite, then it leaves invariant a positive definite Hermitian form g and thus any linear combination of g and h. However, g and h can be simultaneously diagonalized, so that for some t in $\mathbb{C}$ the form $f := g - th$ is degenerate but not zero. But then U leaves invariant the radical of f, contradicting the irreducibility.

The representation given by (10) is the contragredient representation of U given by (7); that is, strictly speaking it is a representation on the dual of $\mathbb{C}^n$. The advantage of (10) over (7) is that the reflexion hyperplanes of the R_k's are in general position whether or not h is non-degenerate. Abstractly the two representations give the same group, with the generators R_k corresponding to each other.

Let h be positive semi-definite and let its radical be 1-dimensional. Since $\Delta := \det(a_{ij}) = 0$, there exist constants $z_1, ..., z_n$, not all 0, such that

$$\sum_{j=1}^{n} a_{jk} z_j = 0 \qquad (k = 1, ..., n).$$

Therefore the reflexion R_k given by (10) transforms the covariant coordinate vector $(x^1, ..., x^n)$ into $(y^1, ..., y^n)$ such that

$$\sum_{j=1}^{n} z_j y^j = \sum_{j=1}^{n} z_j x^j.$$

This shows that we can regard $U = \langle R_1, ..., R_n \rangle$ as operating on the affine $(n-1)$-space

$$\sum_{j=1}^{n} z_j x^j = 1,$$

in which the semi-definite form h induces a unitary metric; in this space, the R_k's appear as reflexions in the walls of an $(n-1)$-simplex formed by the sections of the hyperplanes $x^k = 0$ (cf. [6, p. 257]). This way U becomes an infinite unitary reflexion group in dimension $n-1$.

In [6] the identification of the abstract groups (5) with certain unitary groups is made by studying special kinds of Hermitian forms h and corresponding reflexion groups $U = \langle R_1, ..., R_n \rangle$. Our considerations will follow this approach. For our purpose we need to generalize the construction of [6] to cover some abstract groups which arise naturally in the study of certain universal regular incidence-polytopes.

We write h in the form

$$h(x) = \sum_{i=1}^{n} x_i \bar{x}_i - \frac{1}{2} \cdot \sum_{i \neq j} c_{ij} x_i \bar{x}_j \qquad (c_{ij} = \bar{c}_{ji}), \tag{11}$$

with c_{ij} to be chosen appropriately. If $n = 3$, the three reflexions (7), expressed as matrices, are

$$R_1 = \begin{pmatrix} -1 & c_{21} & c_{31} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ c_{12} & -1 & c_{32} \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_{13} & c_{23} & -1 \end{pmatrix}.$$

To find the period of $R_i R_j$ $(i \neq j)$, we observe that the characteristic equation of $R_i R_j$ is

$$(\lambda - 1)\{(\lambda + 1)^2 - c_{ij} \bar{c}_{ij} \lambda\} = 0.$$

Now, concluding as in [6, pp. 247, 248], the period of $R_i R_j$ is l $(\geqslant 2)$ if $|c_{ij}| = 2\cos(\pi/l)$; hence it is 2 if $c_{ij} = 0$, 3 if $|c_{ij}| = 1$, or ∞ if $c_{ij} = 2$ $(= 2\cos(\pi/\infty))$. Similarly, if two of the numbers c_{ij} have absolute value 1, then the characteristic equation for $R_1 R_2 R_3 R_2$ is

$$(\lambda - 1)\{(\lambda + 1)^2 - (\gamma + 1)(\bar{\gamma} + 1)\,\lambda\} = 0,$$

with

$$\gamma = c_{12} c_{23} c_{31}; \tag{12}$$

hence the period is m if $|\gamma + 1| = 2\cos(\pi/m)$; here $2 \leqslant m \leqslant \infty$. Then, as in [6], if $|\gamma + 1| = 2\cos(\pi/m)$ and two of the numbers c_{12}, c_{23}, c_{13} have absolute value 1 while the other, c_{jk} (say), has absolute value $2\cos(\pi/l)$, then the geometrical group $U = \langle R_1, R_2, R_3 \rangle$ is a quotient of the abstract group $[1\ 1\ 1^l]^m$. If γ takes one of the values $c = c(l, m)$ of Table I above, and thus $|c_{jk}| = |\gamma| = 2\cos(\pi/l)$, then h is positive definite and U is isomorphic to the abstract group $[1\ 1\ 1^l]^m$; the same remains true if $\gamma=\overline{c(l,m)}$. In fact, under these conditions the determinant

$$\Delta=\frac{1}{8}\begin{vmatrix} 2 & -c_{12} & -c_{13} \\ -c_{21} & 2 & -c_{23} \\ -c_{31} & -c_{32} & 2 \end{vmatrix}$$

$$=8-2(|c_{12}|^2+|c_{23}|^2+|c_{31}|^2)-(\gamma+\bar{\gamma})$$

$$=5-4\cos^2(\pi/l)-4\cos^2(\pi/m) \tag{13}$$

takes a positive value.

For simplicity we set $c_m := e^{2\pi i/m}$ for $m \geqslant 2$; then $c_2=-1$. For values $l, m \geqslant 3$ not covered by Table I it seems to be unknown if the geometrical group U and the abstract group W are isomorphic; in any case, $[1\ 1\ 1^l]^m$ is infinite. If one of the parameters l or m is 2, then the other is 3, and

$$[1\ 1\ 1^3]^2=[1\ 1\ 1^2]^3=S_4.$$

Note that for any $l, m \geqslant 2$ we have

$$[1\ 1\ 1^l]^m=[1\ 1\ 1^m]^l$$

(cf. [6, p. 248]).

Note that for infinite unitary groups our notation slightly differs from that of [6]. For example, the infinite unitary group denoted by $[1\ 1\ 1^4]^6$ in [6] satisfies all the relations given by the corresponding diagram (5) but possibly some independent relations too. Hence it is only a quotient of the group we describe by the same symbol.

The Hermitian forms h in 4 or more variables are constructed by imposing certain conditions on some of the 3-dimensional forms h_{pqr} obtained by restricting h to the variables x_p, x_q, and x_r $(1 \leqslant p<q<r \leqslant n)$. This condition is on the coefficients c_{ij} of h_{pqr}, especially on the product $\gamma_{pqr}=c_{pq}c_{qr}c_{rp}$, as above. This will guarantee that the subgroups $\langle R_p, R_q, R_r\rangle$ of the reflexion group $U=\langle R_1, ..., R_n\rangle$ are isomorphic to the corresponding unitary group $[1\ 1\ 1^l]^m$. In all our applications, the index sets $\{p, q, r\}$ will be given by the (marked) triangular 2-faces of the diagram $\mathscr{D}$ on n nodes (in the general sense described above).

For our purpose it suffices to verify the following special case of the intersection property (2) for U:

$$\langle R_i, R_j, R_k\rangle \cap \langle R_i, R_j, R_s\rangle = \langle R_i, R_j\rangle \text{ if } i, j, k, s \text{ are distinct and } \{i, j, k\} \text{ or } \{i, j, s\} \text{ is a 2-face of } \mathscr{D}. \tag{14}$$

To verify (14) observe that by (7) the intersection on the left side fixes both x_k and x_s. On the other hand, if $\{i, j, k\}$ (say) is a 2-face of $\mathscr{D}$, then by construction $\langle R_i, R_j, R_k \rangle$ is a unitary group $[1\ 1\ 1^l]^m$; but then its subgroup fixing the coordinate x_k is just $\langle R_i, R_j \rangle$ (cf. [6, 27]). Note that for $n=4$ property (14) completely covers the intersection property for U.

By construction U belongs to $\mathscr{D}$. For the abstract group $W = W(\mathscr{D})$ and its generators σ_i there is an analogue of (14):

$$\langle \sigma_i, \sigma_j, \sigma_k \rangle \cap \langle \sigma_i, \sigma_j, \sigma_s \rangle = \langle \sigma_i, \sigma_j \rangle \text{ if } i, j, k, s \text{ are distinct and } \{i, j, k\} \text{ or } \{i, j, s\} \text{ is a 2-face of } \mathscr{D}. \tag{15}$$

In fact, $\sigma_i \to R_i$ $(i = 1, ..., n)$ defines a homomorphism $\psi: W \to U$ which is one-to-one on all subgroups $\langle \sigma_i, \sigma_j, \sigma_k \rangle$ for which $\{i, j, k\}$ is a 2-face of $\mathscr{D}$. Note for this that, by construction, $\langle R_i, R_j, R_k \rangle$ is a group $[1\ 1\ 1^l]^m$, while trivially $\langle \sigma_i, \sigma_j, \sigma_k \rangle$ is a quotient of $[1\ 1\ 1^l]^m$. That proves (15).

Let h be indefinite. By Lemma 3, to check non-finiteness of U it suffices to prove irreducibility of U. For $n=4$ this is easily done. In fact, if $\{1, 2, 3\}$ (say) is a 2-face of $\mathscr{D}$, then by (7) the group $\langle R_1, R_2, R_3 \rangle$ $(= [1\ 1\ 1^l]^m)$ acts irreducibly on each of the two subspaces $\{(0, 0, 0, x_4) \mid x_4 \in \mathbb{C}\}$ and $\{(x_1, x_2, x_3, 0) \mid x_1, x_2, x_3 \in \mathbb{C}\}$ of (contravariant) coordinate vectors, but on no others; however, neither of them is invariant under R_4. For larger n we shall usually deduce the non-finiteness of U from that of a suitable subgroup generated by 4 of the reflexions R_i.

For later use we note the following formula for the determinant Δ of h in case $n=4$:

$$\begin{aligned} 16\Delta &= \begin{vmatrix} 2 & -c_{12} & -c_{13} & -c_{14} \\ -c_{21} & 2 & -c_{23} & -c_{24} \\ -c_{31} & -c_{32} & 2 & -c_{34} \\ -c_{41} & -c_{42} & -c_{43} & 2 \end{vmatrix} \\ &= 16 - 4 \cdot \sum_{i<j} |c_{ij}|^2 \\ &\quad + (|c_{12}|^2\,|c_{34}|^2 + |c_{13}|^2\,|c_{24}|^2 + |c_{14}|^2\,|c_{23}|^2) \\ &\quad - 2 \cdot \sum_{i<j<k} (\gamma_{ijk} + \bar{\gamma}_{ijk}) - \sum_{\substack{i,j,k \neq 1 \\ \text{distinct}}} \delta_{1ijk}, \end{aligned} \tag{16}$$

with $\gamma_{ijk} := c_{ij} c_{jk} c_{ki}$ and $\delta_{ijks} := c_{ij} c_{jk} c_{ks} c_{si}$ for distinct i, j, k, s. Note that $\gamma_{ijk} = \bar{\gamma}_{ikj}$ and $\delta_{ijks} = \bar{\delta}_{iskj}$ for all i, j, k, s.

4. The Type $\{6, 3, 3\}$

One of the origins of the theory of abstract regular polytopes was a conjecture of Grünbaum [16], according to which every universal regular incidence-polytope (or naturally generated polystroma in his notation)

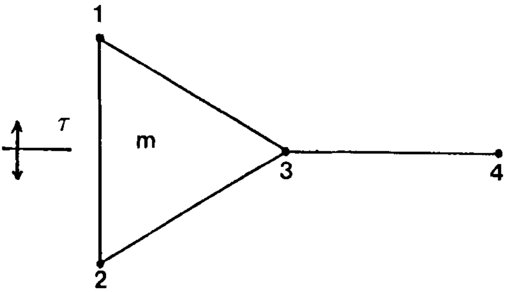
$\mathscr{H}_{b,c} := \{\{6, 3\}_{b,c}, \{3, 3\}\}$, with $b = c \geqslant 2$ or $c = 0$, $b \geqslant 2$, is finite. In this section we disprove this conjecture and prove Theorem 1 below. Amazingly enough it turns out that the only finite instances are precisely those examples which Grünbaum had been able to construct without using groups; see also Weiss [31] for another description of these examples.

THEOREM 1. *$\mathscr{H}_{b,c}$ exists for all (b, c) with $b = c \geqslant 2$ or $c = 0$, $b \geqslant 2$. The only finite instances are $\mathscr{H}_{2,0}$, $\mathscr{H}_{3,0}$, $\mathscr{H}_{4,0}$, and $\mathscr{H}_{2,2}$, that is, the $\mathscr{H}_{b,c}$'s with $b + c \leqslant 4$.*

We discuss the two cases $c = 0$ and $b = c$ separately. Note that the case $(b, c) = (1, 1)$ is dealt with in Section 8.

4.1. *The Universal $\mathscr{H}_{m,0}$*

As remarked in [21], the structure of $\mathscr{H}_{m,0}$ is intimately related to the abstract group $W = [1\ 1\ 2^3]^m$ defined by



(17)

with generators $\sigma_1, \sigma_2, \sigma_3$, and σ_4. In fact, the twisting operation

$$\kappa: (\sigma_1, ..., \sigma_4; \tau) \to (\tau, \sigma_2, \sigma_3, \sigma_4) =: (\rho_0, ..., \rho_3) \tag{18}$$

turns W into a semi-direct product $A = \langle \rho_0, ..., \rho_3 \rangle$ of W by C_2. A presentation of A in terms of $\rho_0, ..., \rho_3$ is given by the standard relations for the Coxeter group $\overset{0}{\bullet}\underset{6}{\text{---}}\overset{1}{\bullet}\text{---}\overset{2}{\bullet}\text{---}\overset{3}{\bullet}$ and the extra relation

$$(\rho_0 \rho_1 \rho_2)^{2m} = 1. \tag{19}$$

This extra relation arises from (4) and

$$(\rho_0 \rho_1 \rho_2)^2 = (\tau \sigma_2 \sigma_3)^2 = \sigma_1 \sigma_3 \sigma_2 \sigma_3.$$

For $m \leqslant 4$ it was already proved in [21] that A is a C-group. In this case W is isomorphic to the unitary reflexion group with the same diagram (17). By (3) and (19) the corresponding regular 4-incidence-polytope is $\mathscr{H}_{m,0}$. For realizations of $\mathscr{H}_{2,0}$, $\mathscr{H}_{3,0}$, and $\mathscr{H}_{4,0}$ see [21, Sect. 10 (b), (a), and (d)].

To discuss the case of general m note that, if the universal $\mathscr{H}_{m,0}$ really exists, then A must be its group. But as remarked in Section 2, for the existence of the universal $\mathscr{H}_{m,0}$ it suffices to find any member $\mathscr{P}$ in $\langle \{6, 3\}_{m,0}, \{3, 3\} \rangle$. We shall construct such a $\mathscr{P}$ by a twisting operation on the (irreducible) reflexion group $U = \langle R_1, ..., R_4 \rangle$ associated with (17) by means of a Hermitian form h.

The form h in 4 variables is given by (11) with

$$c_{12} = c_m (= e^{2\pi i/m}), \qquad c_{13} = c_{23} = c_{34} = 1, \qquad c_{14} = c_{24} = 0. \tag{20}$$

This is precisely the form of [6, pp. 246, 249] associated with (17). By (16), its determinant Δ is given by

$$\Delta = \frac{1}{16}(5 - 8\cos^2(\pi/m)), \tag{21}$$

so that h is positive definite for $m \leqslant 4$ but indefinite for $m > 4$. For the corresponding reflexion group $U = \langle R_1, ..., R_4 \rangle$, the subgroup $\langle R_1, R_2, R_3 \rangle$ is isomorphic to $[1\ 1\ 1]^m$ while $\langle R_1, R_3, R_4 \rangle$ and $\langle R_2, R_3, R_4 \rangle$ are isomorphic to S_4. Thus U belongs to the diagram (17), but may not be isomorphic to the abstract group W if $m > 4$. We conjecture that U and W are actually isomorphic for all m. Since U acts irreducibly on $\mathbb{C}^4$, it follows from Lemma 3 that U (and thus W) is infinite if $m > 4$. Also, (14) and (15) imply that both U and W have the intersection property with respect to their generators. By the construction of A this already implies that A is a C-group, so that $\mathscr{H}_{m,0}$ exists for all m and is infinite for $m > 4$. This suffices to prove the corresponding part of Theorem 1.

However, we can go further and use U to construct an incidence-polytope $\mathscr{P}$ in $\langle \{6, 3\}_{m,0}, \{3, 3\} \rangle$ which also admits a geometrical realization; if the above conjecture is true, then $\mathscr{P} = \mathscr{H}_{m,0}$.

To apply our methods observe first that U also admits an involutory outer automorphism T interchanging the generators appropriately; see also (60) of Section 7. This can be realized by the semi-linear map

$$T\colon (x_1, ..., x_4) \to (\bar{x}_2, \bar{x}_1, \bar{x}_3, \bar{x}_4). \tag{22}$$

Then

$$\kappa\colon (R_1, ..., R_4; T) \to (T, R_2, R_3, R_4) =: (\alpha_0, ..., \alpha_3) \tag{18'}$$

gives us the semi-direct product $A' = \langle \alpha_0, ..., \alpha_3 \rangle$ of U by C_2. This is a C-group, since U has the intersection property with respect to the R_i's (or by Lemma 2). The corresponding 4-incidence-polytope $\mathscr{P}$ is in $\langle \{6, 3\}_{m,0}, \{3, 3\} \rangle$, and is finite if and only if $m \leqslant 4$.

To construct a (linear) realization for $\mathscr{P}$ we start from the representation of the R_i's given by (7), that is,

$$\begin{cases} R_1: (x_1, ..., x_4) \to (-x_1 + \bar{c}_m x_2 + x_3, x_2, x_3, x_4); \\ R_2: (x_1, ..., x_4) \to (x_1, -x_2 + c_m x_1 + x_3, x_3, x_4); \\ R_3: (x_1, ..., x_4) \to (x_1, x_2, -x_3 + x_1 + x_2 + x_4, x_4); \\ R_4: (x_1, ..., x_4) \to (x_1, x_2, x_3, -x_4 + x_3). \end{cases} \tag{23}$$

The corresponding Wythoff space is (recall our convention!)

$$\alpha_1 \cap \alpha_2 \cap \alpha_3 = R_2 \cap R_3 \cap R_4 = \{\lambda(4, 2 + 3c_m, 4 + 2c_m, 2 + c_m) \mid \lambda \in \mathbb{C}\}.$$

Thus any choice of initial vertex F_0 in this space will give us a complex realization of $\mathscr{P}$. Its "geometrical symmetry group" is A', and thus a group of linear and semi-linear isometries with respect to h. For $m \leqslant 4$ we get unitary realizations, as in [21]. For all m the corresponding real realizations are 8-dimensional, and are invariant under a group of isometries with respect to the real quadratic form induced by h.

Turning to a realization of the dual $\mathscr{P}^*$ of $\mathscr{P}$ we find that now the Wythoff space is

$$\alpha_0 \cap \alpha_1 \cap \alpha_2 = \{s \cdot a + t \cdot b \mid s, t \in \mathbb{R}; 3t = 4s \cdot \sin^2(\pi/m)\},$$

with $a := (2 + c_m)^{-1} (3, 1 + 2c_m, 1, 0)$ and $b := (2 + c_m)^{-1} (-2, -c_m, 0, 1)$. Picking any point in this space gives a complex realization of $\mathscr{P}^*$, from which we get again an 8-dimensional real realization.

Our construction of $\mathscr{H}_{m,0}$ (and $\mathscr{P}$) admits generalizations as follows. First, by starting from the more general group $[1\ 1\ 2^l]^m$ and its diagram (see (5)) similar considerations lead to the classification of the finite universal $\{\{2l, 3\}_{2m}, \{3, 3\}\}$. The corresponding Hermitian form h is defined as in (20) except that the value of c_{12} has to be adjusted appropriately; in particular, $c_{12} = c(l, m)$ if (l, m) occurs in Table I. Except for $\mathscr{H}_{2,0}$, $\mathscr{H}_{3,0}$, and $\mathscr{H}_{4,0}$ the only finite instances are $\{\{2l, 3\}_6, \{3, 3\}\}$ with $l \geqslant 2$; for $l = 2$ this is the 4-cube $\{4, 3, 3\}$, for $l = 3$ it is $\mathscr{H}_{3,0}$.

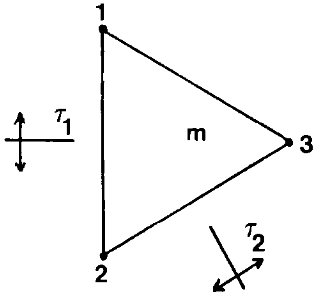
Second, we can construct higher-dimensional regular d-incidence-polytopes $\mathscr{L}_d$ of type $\{2l, 3^{d-2}\}$ $(d \geqslant 4)$ by starting from the group $[1\ 1\ (d-2)^l]^m$. They are universal with respect to having 3-faces of type $\{2l, 3\}_{2m}$. For $d > 4$ the corresponding Hermitian form is obtained from the form for $[1\ 1\ 2^l]^m$ by adding the term

$$(x_5 \bar{x}_5 + \cdots + x_d \bar{x}_d) \\ - \tfrac{1}{2}(x_4 \bar{x}_5 + x_5 \bar{x}_4 + x_5 \bar{x}_6 + x_6 \bar{x}_5 + \cdots + x_{d-1} \bar{x}_d + x_d \bar{x}_{d-1}).$$

Then for $d > 4$ the only finite instances are given by $(l, m) = (3, 2)$ or $l \geqslant 2$, $m = 3$; see [6, p. 250] for the corresponding value of the determinant Δ. If $(l, m) = (3, 2)$, then $[1\ 1\ (d-2)^3]^2 = S_{d+1}$ and $A(\mathscr{L}_d) = S_{d+1} \times C_2$ (cf. [6, p. 250]). If $m = 3$, then $A(\mathscr{L}_d)$ is a semi-direct product of $[1\ 1\ (d-2)^l]^3$ by C_2, of order $2 \cdot l^{d-1} \cdot d!$; in particular, if $l = 2$, then $\mathscr{L}_d$ is the d-cube $\{4, 3^{d-2}\}$. For all d, l, and m, $\mathscr{L}_d = \{\mathscr{L}_{d-1}, \{3^{d-2}\}\}$ (with $\mathscr{L}_3 = \{2l, 3\}_{2m}$).

4.2. *The Universal* $\mathscr{H}_{m,m}$

The structure of $\mathscr{H}_{m,m}$ ($m \geqslant 2$) is more complicated than that of $\mathscr{H}_{m,0}$. We begin by observing that the toroidal map $\{6, 3\}_{m,m}$ can be derived from the group $[1\ \ 1\ \ 1]^m$ with diagram



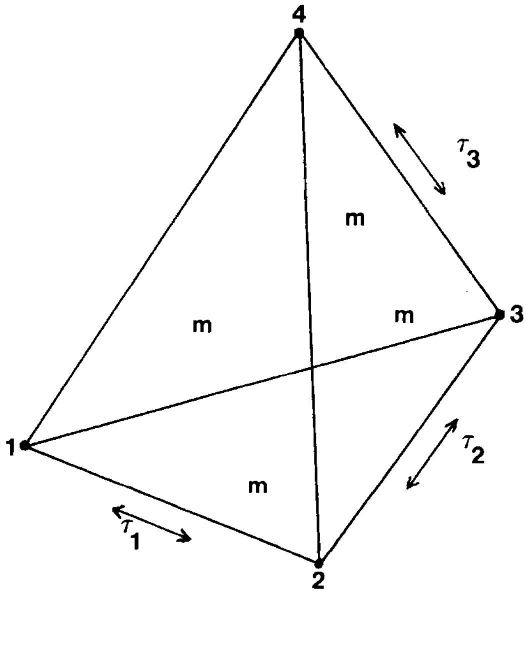
(24)

by means of the operation

$$\kappa\colon (\sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2) \to (\sigma_1, \tau_1, \tau_2) =: (\rho_0, \rho_1, \rho_2). \tag{25}$$

Its group $A = \langle \rho_0, \rho_1, \rho_2 \rangle$ is a semi-direct product of $[1\ \ 1\ \ 1]^m$ by S_3, of order $36m^2$.

For the construction of $\mathscr{H}_{m,m} = \{\{6, 3\}_{m,m}, \{3, 3\}\}$ this suggests studying the abstract group $W = \langle \sigma_1, ..., \sigma_4 \rangle$ with diagram a tetrahedron, all of whose 2-faces are marked m; that is,



(26)

Clearly, the permutations (1 2), (2 3), and (3 4) induce involutory outer automorphisms τ_1, τ_2, and τ_3 of W, respectively.

Now, applying the operation

$$\kappa: (\sigma_1, \ldots, \sigma_4; \tau_1, \ldots, \tau_3) \to (\sigma_1, \tau_1, \tau_2, \tau_3) = (\rho_0, \ldots, \rho_3) \tag{27}$$

gives the group $A = \langle \rho_0, \ldots, \rho_3 \rangle$, a semi-direct product of W by S_4. A presentation for A consists of the standard relations for the Coxeter group $\bullet\underset{6}{\text{———}}\bullet\text{———}\bullet\text{———}\bullet$ and the extra relation

$$(\rho_2(\rho_1\rho_0)^2)^{2m} = 1; \tag{28}$$

this extra relation corresponds to (4). Hence, by (3), if $\mathscr{H}_{m,m}$ exists, then A must be its group. Note that at this point we do not know if the subgroup $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is really isomorphic to $[1\ 1\ 1]^m$, or just to a quotient of this group.

To complete the construction we associate with (26) a Hermitian form h of the form (11) in 4 variables. We choose the c_{ij} in such a way that any three of the corresponding reflexions $R_1, \ldots, R_4$ (defined by (7)) generate a group isomorphic to $[1\ 1\ 1]^m$. By our considerations in Section 3 this is satisfied if $|c_{ij}| = 1$ for $i \neq j$ and each of the four terms γ_{123} $(= c_{12}c_{23}c_{31})$, γ_{124}, γ_{134}, and γ_{234} equals c_m or $\bar{c}_m$. One admissible choice is

$$c_{12} = c_{34} = c_{31} = c_m, \qquad c_{23} = c_{24} = c_{41} = \bar{c}_m. \tag{29}$$

(Another possible choice would be $c_{12} = c_{34} = c_m$, $c_{ij} = 1$ otherwise; this corresponds to the choice in (46). We pick the values of (29), because we shall refer to them again later.) By (16) the determinant Δ of h is given by

$$\begin{aligned} 16\Delta &= -9 - 8(c_m + \bar{c}_m) - (c_m^2 + \bar{c}_m^2) \\ &= -9 - 16\cos(2\pi/m) - 2\cos(4\pi/m). \end{aligned}$$

Thus $\Delta > 0$, $= 0$, or < 0 if $m = 2$, 3, or $\geqslant 4$, respectively. By construction any 3×3-minor determines a positive definite form, so that h itself is positive definite, positive semi-definite, or indefinite if $m = 2$, $m = 3$, or $m \geqslant 4$, respectively.

By (7) the generators for $U = \langle R_1, \ldots, R_4 \rangle$ are given by

$$\left\{\begin{aligned} &R_1: (x_1, \ldots, x_4) \to (-x_1 + \bar{c}_m x_2 + c_m x_3 + \bar{c}_m x_4, x_2, x_3, x_4); \\ &R_2: (x_1, \ldots, x_4) \to (x_1, c_m x_1 - x_2 + c_m x_3 + c_m x_4, x_3, x_4); \\ &R_3: (x_1, \ldots, x_4) \to (x_1, x_2, \bar{c}_m x_1 + \bar{c}_m x_2 - x_3 + \bar{c}_m x_4, x_4); \\ &R_4: (x_1, \ldots, x_4) \to (x_1, x_2, x_3, c_m x_1 + \bar{c}_m x_2 + c_m x_3 - x_4). \end{aligned}\right. \tag{30}$$

For $m \geqslant 3$ the two maps

$$\begin{cases} T_1: (x_1, ..., x_4) \to (c_m \bar{x}_2, c_m \bar{x}_1, \bar{c}_m \bar{x}_3, c_m \bar{x}_4), \\ T_3: (x_1, ..., x_4) \to (\bar{x}_1, c_m^2 \bar{x}_2, \bar{x}_4, \bar{x}_3) \end{cases} \tag{31}$$

provide outer automorphisms of U which correspond to τ_1 and τ_3, respectively; note that $(T_1 T_3)^2 = 1$. Here we do not know if there is also a geometrical analogue T_2 of τ_2; there may not be such a T_2 in $\mathbb{C}^4$, but possibly there is in real 8-space. If $m = 2$ (and thus $c_m = -1$), then $-T_1$ and T_3 of (31) and

$$T_2: (x_1, ..., x_4) \to (\bar{x}_1, \bar{x}_3, \bar{x}_2, \bar{x}_4)$$

give outer automorphisms which generate a group S_4.

In any case, by (15), W has the intersection property (2), so that A is a C-group; alternatively, this follows from Lemma 2. By construction $\langle R_1, R_2, R_3 \rangle$ and $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ are isomorphic to $[1\ 1\ 1]^m$, so that the corresponding regular 4-incidence-polytope is indeed $\mathscr{H}_{m,m}$. Again we conjecture that $U = W$; this is true for $m = 2$.

To construct a regular incidence-polytope as well from U itself we refer to (60) in Section 7. This guarantees the existence of an (abstract) analogue of τ_2 for the (abstract) group U. Hence (27) applies to U as well and gives a polytope in $\langle \{6, 3\}_{m,m}, \{3, 3\} \rangle$.

For $m = 2$ we get the universal $\mathscr{H}_{2,2}$. In this case W is isomorphic to $S_5 = S_{\{1, ..., 5\}}$, with the generators given by $\sigma_i = (i\ 5)$ for $i = 1, ..., 4$ (cf. [6, p. 250]). Here each τ_i can be realized by an inner automorphism of S_5, so that A is isomorphic to $S_5 \times S_4$. For constructions of $\mathscr{H}_{2,2}$ see also [25, 31].

For $m \geqslant 4$ the form h is indefinite, so that U (and thus W) is infinite by Lemma 3. Hence $\mathscr{H}_{m,m}$ is infinite as well.

If $m = 3$, the form h is not definite but only semi-definite. As remarked in Section 3, we can regard U as an infinite unitary group in 3 dimensions, which is, in fact, discrete (cf. [23]). In particular, W and $\mathscr{H}_{3,3}$ are infinite.

We now discuss some realizations. When the associated form h is positive definite, representations in standard orthogonal coordinates ξ_i are available, and lead to the following realizations.

We begin with the map $\{3, 6\}_{m,m}$. The group $[1\ 1\ 1]^m$ can be generated by

$$\begin{cases} R_1: (\xi_1, \xi_2, \xi_3) \to (\xi_1, \xi_3, \xi_2); \\ R_2: (\xi_1, \xi_2, \xi_3) \to (\xi_3, \xi_2, \xi_1); \\ R_3: (\xi_1, \xi_2, \xi_3) \to (c_m \xi_2, \bar{c}_m \xi_1, \xi_3) \end{cases} \tag{32}$$

(cf. [6, p. 259]). As outer automorphisms we can take

$$\begin{cases} T_1: (\xi_1, \xi_2, \xi_3) \to (\bar{\xi}_2, \bar{\xi}_1, \bar{\xi}_3), \\ T_2: (\xi_1, \xi_2, \xi_3) \to \eta(c_m \bar{\xi}_1, \bar{\xi}_3, \bar{\xi}_2), \end{cases} \tag{33}$$

with $\eta := \bar{c}_{3m}$. Then, $(T_1 T_2)^3 = \bar{\eta}^3 \cdot \bar{c}_m \cdot Id = Id$ (=identity map), so that $\langle T_1, T_2 \rangle = S_3$.

For the map $\{6, 3\}_{m,m}$ constructed by (25) the Wythoff space $T_1 \cap T_2$ is $\{0\}$. However, for the dual $\{3, 6\}_{m,m}$ the Wythoff space is

$$R_1 \cap T_1 = \{t(1, 1, 1) \mid t \in \mathbb{R}\}.$$

Then, picking $F_0 = (1, 1, 1)$ as the initial vertex gives us a realization of $\{3, 6\}_{m,m}$ with vertex set the $3m^2$ points

$$\begin{cases} (c_m^\lambda, c_m^\mu, c_m^\nu), & \lambda + \mu + \nu \equiv 0 \pmod{m}; \\ \eta(c_m^\lambda, c_m^\mu, c_m^\nu), & \lambda + \mu + \nu \equiv 1 \pmod{m}; \\ \bar{\eta}(c_m^\lambda, c_m^\mu, c_m^\nu), & \lambda + \mu + \nu \equiv -1 \pmod{m}. \end{cases} \tag{34}$$

Again, all automorphisms are realized by linear and semi-linear (standard) isometries of U_3. The corresponding real euclidean realization is 6-dimensional. We remark that similar realizations for $\{3, 6\}_{m,0}$ were already described in [6, p. 263].

A realization for the dual $\mathscr{H}_{2,2}^* = \{\{3, 3\}, \{3, 6\}_{2,2}\}$ of $\mathscr{H}_{2,2}$ can be derived from the real representation of its group $A = S_5 \times S_4$ given by

$$\begin{cases} R_i: (\xi_1, ..., \xi_9) \to (\xi_1, ..., \xi_{i-1}, \xi_5, \xi_{i+1}, ..., \xi_4, \xi_i; \xi_6, ..., \xi_9) \\ \qquad \text{for} \quad i = 1, ..., 4; \\ T_1: (\xi_1, ..., \xi_9) \to (\xi_2, \xi_1, \xi_3, \xi_4, \xi_5; \xi_7, \xi_6, \xi_8, \xi_9); \\ T_2: (\xi_1, ..., \xi_9) \to (\xi_1, \xi_3, \xi_2, \xi_4, \xi_5; \xi_6, \xi_8, \xi_7, \xi_9); \\ T_3: (\xi_1, ..., \xi_9) \to (\xi_1, \xi_2, \xi_4, \xi_3, \xi_5; \xi_6, \xi_7, \xi_9, \xi_8). \end{cases} \tag{35}$$

Here we regard A as acting (reducibly) only on the 7-dimensional subspace of E^9 defined by

$$\xi_1 + \cdots + \xi_5 = 0, \qquad \xi_6 + \cdots + \xi_9 = 0.$$

Then the Wythoff space $R_1 \cap T_1 \cap T_2$ for $\mathscr{H}_{2,2}^*$ is spanned by a and b, with $a := (1, 1, 1, -4, 1; 0, ..., 0)$ and $b := (0, ..., 0; 1, 1, 1, -3)$. Picking $F_0 = a + b$ as the initial vertex leads to a realization with vertex set the 20 points $(\xi_1, ..., \xi_9)$ with

$$\begin{cases} \xi_i = -4 & \text{for one } i \text{ with } i \leqslant 5; \\ \xi_i = -3 & \text{for one } i \text{ with } i \geqslant 6; \\ \xi_i = 1 & \text{otherwise}. \end{cases} \tag{36}$$

This is in accordance with the fact that $\mathscr{H}^*_{2,2}$ has exactly 20 vertices. Note that this realization is the blend (in the sense of Section 7) of the two degenerate realizations obtained from picking a and b as the initial vertices.

The construction of $\mathscr{H}_{m,m}$ described in this section is really the most interesting special case of a more general construction whose 4-dimensional version is sketched in Section 5.2. For example, as for $\{6, 3\}_{m,0}$, there is a series of regular d-incidence-polytopes $\mathscr{L}_d$ of type $\{6, 3^{d-2}\}$ which are universal with respect to having 3-faces isomorphic to $\{6, 3\}_{m,m}$ $(d \geqslant 3)$. They are constructed from the abstract group W whose diagram is a $(d-1)$-simplex with all 2-faces marked m; then $A(\mathscr{L}_d)$ is a semi-direct product of W by S_d. See Section 5.2 for more details. For $d \geqslant 4$ the only finite instances of $\mathscr{L}_d$ are obtained for $m=2$; then $A(\mathscr{L}_d) = S_{d+1} \times S_d$. For $m=2$ the dual $\mathscr{L}^*_d$ of $\mathscr{L}_d$ admits a geometrical realization in real $(2d-1)$-space which is similar to that of $\mathscr{H}^*_{2,2}$.

5. The Types $\{6, 3, p\}$ with $p = 4$, 5 or 6

In this section we complete the classification of the finite universal locally toroidal regular 4-incidence-polytopes of type $\{6, 3, p\}$. Again, for $\{6, 3\}_{1,1}$ the discussion is postponed to Section 8. For $p=4$ we recall the following result from [20]; see also our Lemma 1.

THEOREM 2. *The universal* $\{\{6, 3\}_{m,0}, \{3, 4\}\}$ *and* $\{\{6, 3\}_{m,m}, \{3, 4\}\}$ *exist for all* $m \geqslant 2$. *The only finite instance is* $\{\{6, 3\}_{2,0}, \{3, 4\}\}$ *whose group is the direct product of* C_2 *and the group* $[3, 3, 4]$ *of the* 4-*crosspolytope* $\{3, 3, 4\}$, *of order* 768.

This result could equally well be obtained by the methods we shall employ for $p=5$ or 6. These methods enable us to prove the following results.

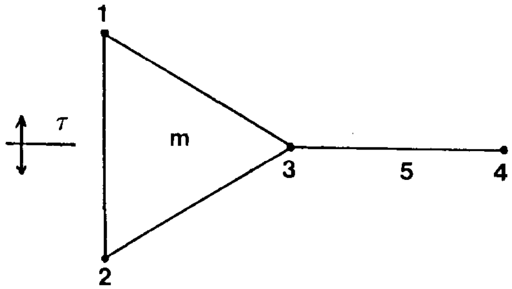
THEOREM 3. *The universal* $\{\{6, 3\}_{m,0}, \{3, 5\}\}$ *and* $\{\{6, 3\}_{m,m}, \{3, 5\}\}$ *exist for all* $m \geqslant 2$. *The only finite instance is* $\{\{6, 3\}_{2,0}, \{3, 5\}\}$ *whose group is the direct product of* C_2 *and the group* $[3, 3, 5]$ *of the* 600-cell $\{3, 3, 5\}$, *of order* 28800.

THEOREM 4. (a) *The universal* $\{\{6, 3\}_{m,m}, \{3, 6\}_{n,0}\}$ *and* $\{\{6, 3\}_{m,m}, \{3, 6\}_{n,n}\}$ *exist for all* $m,n \geqslant 2$. *The only finite instance is* $\{\{6, 3\}_{2,2}, \{3, 6\}_{2,0}\}$, *with group* $S_5 \times S_4 \times C_2$.

(b) *The universal* $\mathscr{M}_{m,n} := \{\{6, 3\}_{m,0}, \{3, 6\}_{n,0}\}$ *exists for all* $m, n \geqslant 2$. *The only finite instances are* $\mathscr{M}_{m,2}$ *and its dual* $\mathscr{M}_{2,m}$ *for* $m \leqslant 4$, *with group a semi-direct product of the unitary group* $[1\ 1\ 2^3]^m$ *by* $C_2 \times C_2$ (*of order* $4 \cdot 5!$, $4 \cdot 3^3 \cdot 4!$, *and* $256 \cdot 5!$ *as* $m = 2$, 3, *or* 4, *respectively*).

5.1. *The Universal* $\{\{6, 3\}_{m,0}, \{3, 5\}\}$

The construction of $\mathscr{H}_{m,0}$ suggests considering the abstract group $W = \langle \sigma_1, ..., \sigma_4 \rangle$ defined by the diagram



(37)

Again (18) turns W into a semi-direct product $A = \langle \rho_0, ..., \rho_3 \rangle$ of W by C_2. A presentation for A in terms of $\rho_0, ..., \rho_3$ is given by the standard relations for the Coxeter group $\bullet\underset{6}{\text{—}}\bullet\text{—}\bullet\underset{5}{\text{—}}\bullet$ and the extra relation (19). Clearly, if the universal $\mathscr{P}_{m,0} := \{\{6, 3\}_{m,0}, \{3, 5\}\}$ exists, then A must be its group.

As before we associate with (37) a Hermitian form h in 4 variables. The c_{ij} are chosen as

$$c_{12} = c_m, \qquad c_{13} = c_{23} = 1, \qquad c_{14} = c_{24} = 0, \qquad c_{34} = \tau, \tag{38}$$

where $\tau = (1 + \sqrt{5})/2 = 2\cos(\pi/5)$ is the golden ratio. Note that the choice of c_{34} guarantees that in the corresponding reflexion group $U = \langle R_1, ..., R_4 \rangle$ given by (7) the product $R_3 R_4$ has order 5. Hence, $\langle R_1, R_3, R_4 \rangle$ and $\langle R_2, R_3, R_4 \rangle$ are isomorphic to the group of the icosahedron $\{3, 5\}$, while $\langle R_1, R_2, R_3 \rangle$ is isomorphic to $[1\ 1\ 1]^m$; the same remains true for the corresponding subgroups of W.

The generators for U are given by

$$\left(\begin{array}{l} R_1\colon (x_1, ..., x_4) \to (-x_1 + \bar{c}_m x_2 + x_3, x_2, x_3, x_4); \\ R_2\colon (x_1, ..., x_4) \to (x_1, -x_2 + c_m x_1 + x_3, x_3, x_4); \\ R_3\colon (x_1, ..., x_4) \to (x_1, x_2, -x_3 + x_1 + x_2 + \tau x_4, x_4); \\ R_4\colon (x_1, ..., x_4) \to (x_1, x_2, x_3, -x_4 + \tau x_3). \end{array}\right. \tag{39}$$

Choosing the outer automorphism

$$T\colon (x_1, ..., x_4) \to (\bar{x}_2, \bar{x}_1, \bar{x}_3, \bar{x}_4)$$

and applying the operation (18′) gives a semi-direct product A' of U by C_2; see also (60) in Section 7. Because of (14) this is a C-group; the resulting 4-incidence-polytope $\mathscr{P}$ belongs to $\langle \{6, 3\}_{m,0}, \{3, 5\} \rangle$. As a consequence, the universal $\mathscr{P}_{m,0}$ exists.

To decide finiteness or non-finiteness of $\mathscr{P}_{m,0}$ we compute the determinant Δ of h. By (16),

$$16\Delta = 5 - 3\tau - 8\cos^2(\pi/m).$$

This shows that h is positive definite for $m=2$ but indefinite for $m \geqslant 3$. Hence, $\mathscr{P}$ and $\mathscr{P}_{m,0}$ are infinite if $m \geqslant 3$.

From [4] we know that the group order of $\mathscr{P}_{2,0}$ is 28800. By Lemma 1, $A(\mathscr{P}_{2,0}) = [3, 3, 5] \times C_2$. Alternatively, if $\varphi_0, ..., \varphi_3$ are the distinguished generators for $[3, 3, 5]$, then

$$\sigma_1 \to \varphi_0\varphi_1\varphi_0, \qquad \sigma_i \to \varphi_{i-1} \qquad (i = 2, 3, 4) \tag{40}$$

defines the isomorphism between W (with $m=2$) and $[3, 3, 5]$. Since in $[3, 3, 5]$ conjugation by φ_0 keeps φ_2 and φ_3 fixed but interchanges φ_1 and $\varphi_0\varphi_1\varphi_0$, the outer automorphism τ is realizable by an inner automorphism, so that $A(\mathscr{P}_{2,0}) = [3, 3, 5] \times C_2$. The group U is unitary, since h is positive definite for $m=2$. Hence, as a quotient of $W = [3, 3, 5]$ it must be W itself; thus $\mathscr{P} = \mathscr{P}_{2,0}$ if $m=2$.

A 5-dimensional euclidean realization of $\mathscr{P}_{2,0}$ can be obtained from the natural realization of $\{3, 3, 5\}$ in E^4 as follows. Represent $\varphi_0, ..., \varphi_3$ by hyperplane reflexions in $E^5 = E^4 \times E$ such that their restrictions to $E^4 \times \{0\}$ are precisely the distinguished generating hyperplane reflexions for $\{3, 3, 5\}$ in 4-space. By (40) this gives a representation for the σ_i's. Now let τ be the composite of φ_0 followed by the reflexion in the hyperplane $\xi_5 = 0$ of E^5. Then the Wythoff space

$$\rho_1 \cap \rho_2 \cap \rho_3 = \varphi_1 \cap \varphi_2 \cap \varphi_3$$

for $\mathscr{P}_{2,0}$ is a 2-dimensional plane in E^5. If we choose in this plane an initial vertex F_0 which is not in the hyperplane $\xi_5 = 0$, then we get a realization of $\mathscr{P}_{2,0}$ with twice as many vertices as $\{3, 3, 5\}$, that is, with 240 vertices. Note that we cannot expect a faithful realization for the dual $\mathscr{P}^*_{2,0} = \{\{5, 3\}, \{3, 6\}_{2,0}\}$ of $\mathscr{P}_{2,0}$, since $\{3, 6\}_{2,0}$ does not have such a realization (cf. [18]).

Observe that as in Section 4.1 we can find non-trivial realizations for both $\mathscr{P}$ and its dual $\mathscr{P}^*$ for any $m \geqslant 2$. In fact, in both cases the Wythoff space is non-trivial.

5.2. *The Universal* $\{\{6, 3\}_{m,m}, \mathscr{M}\}$

As remarked at the end of Section 4.2, our construction of $\mathscr{H}_{m,m}$ is the most interesting case of a more general construction which we sketch below. As a further application it yields the classification for the universal $\{\{6, 3\}_{m,m}, \{3, 5\}\}$, $m \geqslant 2$. The construction resembles the construction of the polytopes $2^{\mathscr{M},\mathscr{G}}$ of [20] except that now W is not a Coxeter group but a more complicated group.

Let $\mathcal{M}$ be a finite map of type $\{3, p\}$ which is a lattice. This assumption covers almost all interesting cases for $\mathcal{M}$ but excludes, for example, the map $\{3, 6\}_{2,0}$; this map is covered by Lemma 1. Let v be the number of vertices of $\mathcal{M}$.

Consider the following diagram $\mathcal{D}$ whose set of nodes, branches, and triangles is the set of vertices, edges, and 2-faces of the $(v-1)$-simplex, respectively; for convenience we index the nodes of $\mathcal{D}$ by the vertices of $\mathcal{M}$. The branches of $\mathcal{D}$ not corresponding to edges of $\mathcal{M}$ are labelled by ∞, while the branches of $\mathcal{D}$ corresponding to edges of $\mathcal{M}$ remain unlabelled (that is, are labelled by 3). Further, we mark a triangle of $\mathcal{D}$ by m if it corresponds to a 2-face of $\mathcal{M}$. The other triangles of $\mathcal{D}$ remain unmarked, that is, are marked by ∞.

This diagram is regarded as representing the abstract group $W = W(\mathcal{D})$ with generators σ_F, one for each vertex F of $\mathcal{M}$, and a set of extra defining relations of the form (4), one for each marked triangle of $\mathcal{D}$. By construction $A(\mathcal{M})$ acts on W as a group of automorphisms permuting the generators σ_F. The construction in 4.2 corresponds to the case $\mathcal{M} = \{3, 3\}$.

Now, if τ_0, τ_1, τ_2 are the distinguished generators of $A(\mathcal{M})$ defined with respect to some base flag $\{F_{-1}, F_0, ..., F_3\}$ of $\mathcal{M}$, then the operation

$$\kappa : (\{\sigma_F\}_F; \tau_0, \tau_1, \tau_2) \to (\sigma_{F_0}, \tau_0, \tau_1, \tau_2) =: (\rho_0, ..., \rho_3) \tag{41}$$

gives us the semi-direct product $A = \langle \rho_0, ..., \rho_3 \rangle$ of W by $A(\mathcal{M})$. We shall show that A is a C-group and that the corresponding regular 4-incidence-polytope $\mathcal{L}'$ is in $\langle \{6, 3\}_{m,m}, \mathcal{M} \rangle$.

Again we associate with $\mathcal{D}$ a Hermitian form h whose variables are indexed by the vertices of $\mathcal{M}$. For the branches $\{F, G\}$ of $\mathcal{D}$ not corresponding to edges of $\mathcal{M}$ we define the coefficient c_{FG} of h by $c_{FG} := 2$ $(= 2\cos(\pi/l)$ if $l = \infty)$. For the branches $\{F, G\}$ of $\mathcal{D}$ which correspond to edges of $\mathcal{M}$ we choose the coefficients c_{FG} as c_m or $\bar{c}_m$ subject to the restriction that

$$\gamma_{FGH} := \left(\begin{array}{l} c_{FG} c_{GH} c_{HF} = c_m \text{ or } \bar{c}_m \\ \text{if } \{F, G, H\} \text{ corresponds to a 2-face of } \mathcal{M}. \end{array} \right. \tag{42}$$

Such a choice is always possible; for example, for some fixed vertex F define $c_{FG} := 2$ or c_m for all branches $\{F, G\}$, and proceed inductively with the induced diagram on the $v-1$ nodes distinct from F.

The corresponding reflexion group

$$U = \langle R_F \mid F \text{ a vertex of } \mathcal{M} \rangle$$

is a quotient of W. By (42), if $\{F, G, H\}$ corresponds to a 2-face of $\mathcal{M}$, then both $\langle R_F, R_G, R_H \rangle$ and $\langle \sigma_F, \sigma_G, \sigma_H \rangle$ are isomorphic to $[1\ 1\ 1]^m$.

However, it seems that the homomorphism from W onto U need not be one-to-one when restricted to subgroups $\langle \sigma_F, \sigma_G, \sigma_H \rangle$, for which $\{F, G, H\}$ does not correspond to a 2-face of $\mathcal{M}$. In fact, if the edge graph of $\mathcal{M}$ contains a 3-cycle not bounding a 2-face, with vertices F, G, H (say), then the above choice of the coefficients implies $\gamma_{FGH} = c_m, \bar{c}_m, c_{3m}$ or $\bar{c}_{3m}$, so that $\langle R_F, R_G, R_H \rangle$ is isomorphic to $[1\ 1\ 1]^m$ or $[1\ 1\ 1]^{3m}$, but not to $[1\ 1\ 1]$ $(=[1\ 1\ 1]^{\infty})$; on the other hand, it is likely that $\langle \sigma_F, \sigma_G, \sigma_H \rangle$ is really $[1\ 1\ 1]$.

The intersection property (2) for A follows directly from the structure of A as a semi-direct product of W by $A(\mathcal{M})$; in fact, if $F = F_0, G, H$ are the vertices of the 2-face F_2 in the base flag of $\mathcal{M}$, then

$$\begin{aligned}\langle \rho_0, \rho_1, \rho_2 \rangle \cap \langle \rho_1, \rho_2, \rho_3 \rangle &= (\langle \sigma_F, \sigma_G, \sigma_H \rangle \cdot \langle \tau_0, \tau_1 \rangle) \cap \langle \tau_0, \tau_1, \tau_2 \rangle \\ &= \langle \tau_0, \tau_1 \rangle = \langle \rho_1, \rho_2 \rangle.\end{aligned}$$

This proves the existence of $\mathcal{L}'$, and thus of the universal $\mathcal{L}$. From (60) in Section 7 we know that the (abstract) group U admits $A(\mathcal{M})$ as a group of (abstract) outer automorphisms. It follows that U itself can be used to construct a regular incidence-polytope in $\langle \{6, 3\}_{m,m}, \mathcal{M} \rangle$; note that the intersection property for the corrsponding group follows as for A.

Now, U can be used to decide finiteness or non-finiteness of W and $\mathcal{L}$. By construction, if a branch $\{F, G\}$ of $\mathcal{D}$ does not correspond to an edge of $\mathcal{M}$, then the products $R_F R_G$ in U and $\sigma_F \sigma_G$ in W have infinite order. As a consequence, U, W, and $\mathcal{L}$ can only be finite if $\mathcal{M}$ is *neighbourly*, that is, if any two vertices of $\mathcal{M}$ are joined by an edge. However, it is not hard to see that under our assumptions $\mathcal{M}$ can only be neighbourly if it is the tetrahedron $\{3, 3\}$ or the hemi-icosahedron $\{3, 5\}/2$; the first case was dealt with in Section 4.2.

Let $\mathcal{M} = \{3, 5\}/2$. Among the various choices for the c_{FG}'s we can pick one in which the "induced choice" for the subdiagram on the 4 vertices F, G, H, K (say) of two adjacent 2-faces of $\mathcal{M}$ is (up to relabelling) that of (29); note that this is an example where in a sense marks on non-faces are implicitly introduced. Then, restricting the Hermitian form h to the corresponding 4 variables x_F, x_G, x_H, x_K we obtain a form g which is positive definite, positive semi-definite, or indefinite if $m = 2$, $m = 3$, or $m \geqslant 4$, respectively. In any case, if $m \geqslant 3$, then the subgroup $\langle R_F, R_G, R_H, R_K \rangle$ of U is infinite, and thus so are W and $\mathcal{L}$.

The case $m = 2$ needs special discussion. Here the abstract group W is isomorphic to $S_{v+1} = S_7$; if Z is any element which is not a vertex of $\mathcal{M}$, then the transpositions $\sigma_F = (F\ Z)$ are appropriate generators (cf. [9, p. 64, presentation (6.28)]). This proves that h must be positive definite for $m = 2$. The 4-incidence-polytope $\mathcal{L}'$ belongs to $\langle \{6, 3\}_{2,2}, \{3, 5\}/2 \rangle$; since the outer automorphisms τ of $W = S_7$ can be realized by conjugation in S_7, the group of $\mathscr{L}'$ is the direct product of S_7 and A_5 $(=A(\{3,5\}/2))$. Using the Coxeter–Todd algorithm Asia Ivic Weiss has checked on a computer that $\mathscr{L}'$ is not universal. In fact, the coset enumeration seems to indicate that the universal $\mathscr{L}$ is infinite (private communication).

We summarize the results in the next theorem, which also has an appropriate higher-dimensional analogue.

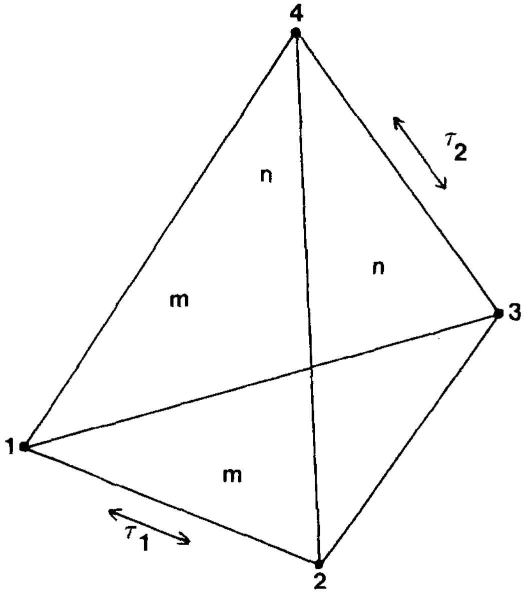
THEOREM 5. *Let $\mathscr{M}$ be a finite regular map of type $\{3, p\}$ which is a lattice. Then the universal $\{\{6,3\}_{m,m}, \mathscr{M}\}$ exists for all $m \geqslant 2$. The only finite instances are $\mathscr{H}_{2,2} = \{\{6,3\}_{2,2}, \{3,3\}\}$ and possibly $\{\{6,3\}_{2,2}, \{3,5\}/2\}$.*

We remark that a suitable variant of our construction with $m=2$ leads to the construction underlying [25, Theorem 2].

5.3. *The Type* $\{6, 3, 6\}$

For the classification of the type $\{6,3,6\}$ three cases have to be discussed; again, facets $\{6,3\}_{1,1}$ and vertex-figures $\{3,6\}_{1,1}$ are excluded (see Section 8). The first two, $\{\{6,3\}_{m,m}, \{3,6\}_{n,0}\}$ and $\{\{6,3\}_{m,m}, \{3,6\}_{n,n}\}$, are already covered by Theorem 5 and Lemma 1. These universals exist for all m and n but the only finite instance is $\{\{6,3\}_{2,2}, \{3,6\}_{2,0}\}$, with group $S_5 \times S_4 \times C_2$. Also, for all m, the group of $\{\{6,3\}_{m,m}, \{3,6\}_{2,0}\}$ is $A(\mathscr{H}_{m,m}) \times C_2$. It remains to consider the third case, which is $\mathscr{M}_{m,n} := \{\{6,3\}_{m,0}, \{3,6\}_{n,0}\} = \{\{6,3\}_{2m}, \{3,6\}_{2n}\}$ (cf. [10]).

Consider the abstract group $W = \langle \sigma_1, ..., \sigma_4 \rangle$ defined by the tetrahedral diagram



(43)

Here the 2-faces of the tetrahedron are marked m and n, the faces $\{1, 2, 3\}$ and $\{1, 2, 4\}$ by m, and the faces $\{1, 3, 4\}$ and $\{2, 3, 4\}$ by n. Again, each mark represents the corresponding relation (4). By construction, W admits two outer automorphisms τ_1 and τ_2 corresponding to the permutations $(1\ 2)$ and $(3\ 4)$, respectively.

Now, the operation

$$\kappa: (\sigma_1, ..., \sigma_4; \tau_1, \tau_2) \to (\tau_1, \sigma_2, \sigma_3, \tau_2) =: (\rho_0, ..., \rho_3) \tag{44}$$

gives the semi-direct product $A = \langle \rho_0, ..., \rho_3 \rangle$ of W by $C_2 \times C_2$. A presentation for A is given by the standard relations for the Coxeter group $\bullet\underset{6}{\text{---}}\bullet\underset{3}{\text{---}}\bullet\underset{6}{\text{---}}\bullet$ and the two extra relations

$$(\rho_0 \rho_1 \rho_2)^{2m} = (\rho_1 \rho_2 \rho_3)^{2n} = 1; \tag{45}$$

these extra relations correspond to (4). Hence, by (3), if $\mathcal{M}_{m,n}$ exists, then A must be its group.

We proceed by associating with (43) a Hermitian form h in 4 variables. The c_{ij} are chosen according to

$$c_{12} = c_m, \qquad c_{13} = c_{14} = c_{23} = c_{24} = 1, \qquad c_{34} = c_n, \tag{46}$$

so that

$$c_{12} c_{23} c_{31} = c_{12} c_{24} c_{41} = c_m, \qquad c_{23} c_{34} c_{42} = c_{13} c_{34} c_{41} = c_n.$$

Consequently, for the corresponding group $U = \langle R_1, ..., R_4 \rangle$ defined by (7) the subgroups $\langle R_1, R_2, R_3 \rangle$ and $\langle R_2, R_3, R_4 \rangle$ are isomorphic to $[1\ 1\ 1]^m$ and $[1\ 1\ 1]^n$, respectively. This remains true for the corresponding subgroups of W. Also, by (14) and (15), U and W have the intersection property (2). It follows immediately that A is a C-group, or equivalently, that $\mathcal{M}_{m,n}$ exists.

The generators of U are given by

$$\begin{cases} R_1: (x_1, ..., x_4) \to (-x_1 + \bar{c}_m x_2 + x_3 + x_4, x_2, x_3, x_4); \\ R_2: (x_1, ..., x_4) \to (x_1, -x_2 + c_m x_1 + x_3 + x_4, x_3, x_4); \\ R_3: (x_1, ..., x_4) \to (x_1, x_2, -x_3 + x_1 + x_2 + \bar{c}_n x_4, x_4); \\ R_4: (x_1, ..., x_4) \to (x_1, x_2, x_3, -x_4 + x_1 + x_2 + c_n x_3). \end{cases} \tag{47}$$

We have not been able to find geometric outer automorphisms T_1 and T_2 for U which correspond to τ_1 and τ_2, either in $\mathbb{C}^4$ or in E^8. However, U admits the automorphism

$$(x_1, ..., x_4) \to (\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3)$$

which would correspond to the half-turn $T_1 T_2$. On the other hand, by (60) in Section 7 we know that abstractly U admits two outer automorphisms corresponding to τ_1 and τ_2. Then, the analogue of (44) applied to U gives a regular 4-incidence-polytope in the class $\langle \{6, 3\}_{m,0}, \{3, 6\}_{n,0} \rangle$.

By (16) the determinant Δ of h is given by

$$\left\{\begin{aligned} 16\Delta &= -7 - 8\cos(2\pi/m) - 8\cos(2\pi/n) - 4\cos(2\pi/m)\cos(2\pi/n) \\ &= 9 - 4(2+\cos(2\pi/m))(2+\cos(2\pi/n)) \\ &= 9 - 4(1 + 2\cos^2(\pi/m))(1+2\cos^2(\pi/n)). \end{aligned}\right. \tag{48}$$

This shows that h is

$$\left\{\begin{array}{ll} \text{positive definite} & \text{if } (m, n) = (2, 2), (2, 3), (2, 4), (3, 2), (4, 2); \\ \text{positive semi-definite} & \text{if } (m, n) = (3, 3); \\ \text{indefinite} & \text{otherwise.} \end{array}\right. \tag{49}$$

It follows that U, and thus $\mathcal{U}_{m,n}$, is infinite if $m + n > 6$. For $(m, n) = (3, 3)$ the form is positive semi-definite, so that via the contragredient representation U can be regarded as an infinite unitary reflexion group in 3 dimensions. In particular, $\mathcal{U}_{3,3}$ is infinite as well.

In the remaining cases where $m = 2$ or $n = 2$ the form is positive definite, so that U becomes a unitary group. The following considerations show that U is finite and isomorphic to W.

Let $n = 2$ (say). By Lemma 1 the universal $\mathcal{U}_{m,2}$ is related to $\mathcal{H}_{m,0}$ by $A = A(\mathcal{U}_{m,2}) = A(\mathcal{H}_{m,0}) \times C_2$. Then $\mathcal{U}_{m,2}$ is finite if and only if $m \leqslant 4$. Hence, if $m \leqslant 4$, then W and U must be finite.

Let $m \leqslant 4$. The connexion of $\mathcal{U}_{m,2}$ with $\mathcal{H}_{m,0}$ suggests that U is isomorphic to the unitary group $[1\ 1\ 2^3]^m$ defined by (17). To establish the isomorphism, change the generators $R_1, ..., R_4$ of U to the new generators

$$S_i = R_i \qquad (i = 1, 2, 3), \qquad S_4 = R_4 R_3 R_4. \tag{50}$$

It is easily checked that $S_1, ..., S_4$ satisfy the defining relations for $[1\ 1\ 2^3]^m$. Since both groups are unitary reflexion groups and $\langle R_1, R_2, R_3 \rangle = [1\ 1\ 1]^m$, the isomorphism follows from the classification of the finite unitary reflexion groups (cf. [6]). Hence, $A(\mathcal{U}_{m,2})$ is a semi-direct product of U by $C_2 \times C_2$. In particular, $W = U \cong [1\ 1\ 2^3]^m$. This completes the proof of Theorem 4.

Concluding this section we remark on a generalization of the construction. If the two edges $\{1, 2\}$ and $\{3, 4\}$ of (43) are marked by l and p (say), respectively, then the same operation (44) applied to the corresponding abstract group W gives a group $A = \langle \rho_0, ..., \rho_3 \rangle$ which is the group of the universal $\mathcal{P} = \{\{2l, 3\}_{2m}, \{3, 2p\}_{2n}\}$, at least in the cases where the groups $[1\ 1\ 1^l]^m$ and $[1\ 1\ 1^p]^n$ are finite; see Table I. In fact, a presentation for A is given by the standard relations for the Coxeter group $\overset{0}{\bullet} \overset{}{\underset{2l}{-\!-}} \overset{1}{\bullet} -\!- \overset{2}{\bullet} \overset{}{\underset{2p}{-\!-}} \overset{3}{\bullet}$ and the extra relations (45); the latter define the facets and vertex-figures of $\mathcal{P}$ (cf. [10, p. 111]). Let us assume that $l, p \neq 2$ and $(l, p) \neq (3, 3)$, since these choices are covered by Theorems 2 and 4.

Now, the choice of the c_{ij} in the corresponding Hermitian form h is essentially as in (46) except that c_m and c_n have to be replaced by the appropriate numbers $c(l, m)$ and $c(p, n)$ of Table I, respectively. Then, by (16),

$$\begin{aligned} 16\Delta = &-4(1+c_{12})(1+\bar{c}_{12}) - 4(1+c_{34})(1+\bar{c}_{34}) \\ &-(c_{12}+\bar{c}_{12})(c_{34}+\bar{c}_{34}) + c_{12}\bar{c}_{12}c_{34}\bar{c}_{34} + 8, \end{aligned} \tag{51}$$

with $c_{12} = c(l, m)$, $c_{34} = c(p, n)$. Inserting all possible values we find that h is

$$\begin{cases} \text{positive definite} \\ \qquad \text{if} \quad (l, m, p, n) = (3, 2, s, 3) \quad \text{or} \quad (s, 3, 3, 2) \quad \text{with} \quad s \geqslant 4; \\ \text{positive semi-definite} \\ \qquad \text{if} \quad (l, m, p, n) = \begin{cases} (3, 2, 4, 4) \quad \text{or} \quad (4, 4, 3, 2); \\ (s, 3, t, 3) \quad \text{with} \quad s, t \geqslant 3, (s, t) \neq (3, 3); \end{cases} \\ \text{indefinite} \qquad \text{otherwise}. \end{cases} \tag{52}$$

The positive definite case gives the finite universal $\mathcal{P} = \{\{2l, 3\}_6, \{3, 6\}_{2,0}\}$ and its dual. By Lemma 1, $\mathcal{P}$ and $\mathcal{L} := \{\{2l, 3\}_6, \{3, 3\}\}$ are related by $A(\mathcal{P}) = A(\mathcal{L}) \times C_2$. In Section 4.1, the group $A(\mathcal{L})$ was recognized as a semi-direct product of the unitary group $[1\ 1\ 2^l]^3$ by C_2, so that $A(\mathcal{P})$ is a semi-direct product of $[1\ 1\ 2^l]^3$ by $C_2 \times C_2$, of order $96l^3$. This suggests that the reflexion group U belonging to h (which is now an unitary group) is isomorphic to $[1\ 1\ 2^l]^3$. Again this can be proved as in (50). Also, $W = U = [1\ 1\ 2^l]^3$.

In the positive semi-definite case we can again regard U as an infinite unitary reflexion group in 3 dimensions. We remark that in the case $(l, m, p, n) = (s, 3, t, 3)$ this group is discrete if and only if

$$(s, t) = \begin{cases} (2, 2), (2, 3), (3, 2), (2, 6), (6, 2), (3, 6), (6, 3), \\ (4, 4), (6, 6) \qquad (\text{or} \quad (3, 3)). \end{cases}$$

In any case, the corresponding polytopes $\{\{6, 3\}_{2,0}, \{3, 8\}_8\}$, $\{\{8, 3\}_8, \{3, 6\}_{2,0}\}$, and $\{\{2l, 3\}_6, \{3, 2p\}_6\}$ are infinite. Note that the first two incidence-polytopes are related to the infinite $\{\{3, 3\}, \{3, 8\}_8\}$ and its dual, so that their group is a semi-direct product of the 3-dimensional discrete unitary group $[1\ 1\ 2^4]^4$ by $C_2 \times C_2$.

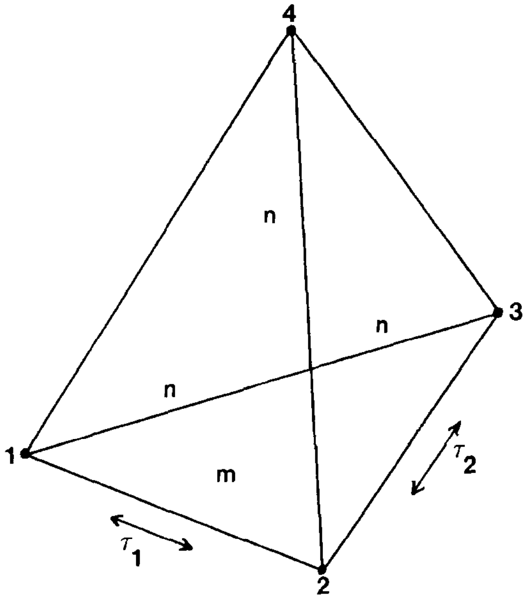
In the indefinite case the groups U and W are infinite, and so is the universal $\mathscr{P}$.

6. The Type $\{3, 6, 3\}$

For the type $\{3, 6, 3\}$ we have not been able so far to completely classify the finite universal regular incidence-polytopes. Here our method seems to fail except for the case of the polytopes $\mathscr{R}_{m,n} := \{\{3, 6\}_{m,m}, \{6, 3\}_{n,0}\}$ with $n = m \geqslant 2$ or $n = 3m \geqslant 6$.

Theorem 6. *The universal* $\{\{3, 6\}_{m,m}, \{6, 3\}_{m,0}\}$ *and* $\{\{3, 6\}_{m,m}, \{6, 3\}_{3m,0}\}$ *exist for all* $m \geqslant 2$. *The only finite instance is* $\{\{3, 6\}_{2,2}, \{6, 3\}_{2,0}\}$, *with group* $S_5 \times S_3$.

To prove the theorem let m, n ($\geqslant 2$) be arbitrary. Consider the abstract group $W = \langle \sigma_1, \ldots, \sigma_4 \rangle$ defined by the tetrahedral diagram



(53)

Here the 2-face $\{1, 2, 3\}$ is marked m while all the others are marked n. The outer automorphisms τ_1 and τ_2 correspond to the permutations $(1\ 2)$ and $(2\ 3)$, respectively.

Now, the operation

$$\kappa\colon (\sigma_1, \ldots, \sigma_4; \tau_1, \tau_2) \to (\tau_1, \tau_2, \sigma_3, \sigma_4) =: (\rho_0, \ldots, \rho_3) \tag{54}$$

gives the group $A = \langle \rho_0, \ldots, \rho_3 \rangle$ which is a semi-direct product of W by S_3. This group has a presentation consisting of the standard relations for the Coxeter group $\bullet\!-\!\!-\!\bullet\underset{6}{-\!\!-}\bullet\!-\!\!-\!\bullet$ and the two extra relations

$$(\rho_0(\rho_1\rho_2)^2)^{2m} = (\rho_1\rho_2\rho_3)^{2n} = 1 \tag{55}$$

which correspond to (4). By (3), if $\mathscr{R}_{m,n}$ exists, then A is its group.

Again, to check the properties of A we associate with (53) a Hermitian form h in 4 variables. Here the c_{ij} must be chosen such that the product $c_{12}c_{23}c_{31}$ is c_m or $\bar{c}_m$ while the products $c_{21}c_{14}c_{42}$, $c_{41}c_{13}c_{34}$, and $c_{43}c_{32}c_{24}$ are c_n or $\bar{c}_n$. Since the product of the four products must be 1, this restricts the parameters m and n. In fact, necessarily $n = m$ or $n = 3m$, which we assume from now on.

If $n = m$, the underlying group W is the same as in (26), and so is the Hermitian form h and the reflexion group U; see (29), (30), (31). As we saw, U and W have the intersection property (2). This immediately proves that A is a C-group. Since $\langle R_1, R_2, R_3 \rangle$ and $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ are isomorphic to $[1\ 1\ 1]^m$, the corresponding 4-incidence-polytope is $\mathscr{R}_{m,m}$. We conjecture that $U = W$ in this case. Again, by (60) in Section 7, we can also construct a regular 4-incidence-polytope by applying an analogue of (54) to U.

If $n = m \geqslant 3$, then $\mathscr{R}_{m,m}$ is infinite. An interesting special case is $n = m = 3$, since then U can be regarded as an infinite unitary reflexion group in 3 dimensions.

Let $n = m = 2$. Then the universal $\mathscr{R}_{2,2} = \{\{3, 6\}_{2,2}, \{6, 3\}_{2,0}\}$ is finite, with group $S_5 \times S_3$. Note for this that $W = U = S_5$ (cf. [9, p. 64]).

To find a real 6-dimensional realization for $\mathscr{R}_{2,2}$ we start from a representation of $S_5 \times S_3$ which resembles (35). In fact, if we restrict (35) to the first 8 coordinates and omit T_3, then we can derive a reducible representation on the 6-dimensional subspace

$$\xi_1 + \cdots + \xi_5 = 0, \qquad \xi_6 + \xi_7 + \xi_8 = 0.$$

Then the Wythoff space $T_2 \cap R_3 \cap R_4$ for $\mathscr{R}_{2,2}$ is spanned by a and b, with $a := (-4, 1, \ldots, 1; 0, 0, 0)$ and $b := (0, \ldots, 0; -2, 1, 1)$. With $F_0 := a + b$ as initial vertex we obtain a realization with vertex set the 15 points

$$\begin{cases} \xi_i = -4 & \text{for one } i \text{ with } i \leqslant 5; \\ \xi_i = -2 & \text{for one } i \text{ with } i \geqslant 6; \\ \xi_i = 1 & \text{else.} \end{cases} \tag{56}$$

This realization is the blend of the two degenerate realizations of $\mathscr{R}_{2,2}$ constructed by taking a and b as initial vertices; see Section 7 or [18].

Now, let $n=3m$. Then $c_m=c_n^3$, so that one possible choice for the c_{ij} is given by

$$c_{12}=c_m, \qquad c_{13}=c_{23}=c_{24}=\bar{c}_n, \qquad c_{14}=c_{34}=c_n. \tag{57}$$

By (16),

$$16\Delta = -5-12\cos(2\pi/3m)-6\cos(4\pi/3m)-4\cos(2\pi/m).$$

Then $\Delta<0$ for all m, so that h is indefinite. Therefore, U, W, and A are infinite. By (14) and (15), U and W have the intersection property; in particular, A is a C-group. Since the subgroups $\langle \sigma_i, \sigma_j, \sigma_k \rangle$ are of the right kind, A is indeed the group of the universal $\mathscr{R}_{m,3m}$. In particular, $\mathscr{R}_{m,3m}$ is infinite for all m. Again, we can find an incidence-polytope in $\langle \{3,6\}_{m,m}, \{6,3\}_{3m,0} \rangle$ by starting from U.

We remark that we do not know what the picture is in the case where $n \neq m$, $3m$. The example at the end of Section 8 indicates that there might be cases of $n, m \geqslant 2$, for which $\mathscr{R}_{m,n}$ does not exist. By a computer it was found that $\mathscr{R}_{2,3}$ is finite, with a group of order 41472 $(=8 \cdot |[1\ 1\ 2^6]^3|)$; see [4]. For further partial results on the universal polytopes $\{\{3,6\}_{m,0}, \{6,3\}_{n,0}\}$ and $\{\{3,6\}_{m,m}, \{6,3\}_{n,n}\}$ we refer to [23].

7. Automorphisms of Diagrams

We have run up against the problem of realizing certain abstract involutory automorphisms of diagrams geometrically. We have tried to do this by means of suitable linear or semilinear mappings, but it is clear that these cannot always suffice. (A particular example occurs in Section 5.3 in connexion with (47); if T_1T_2 is semilinear, then either T_1 must be linear and T_2 semilinear, or vice versa, but it seems that neither will work.)

Let us look more closely at what we want to do. We wish to have *some* geometric representation of the abstract group W, and *some* suitable realizations of the automorphisms τ. But we are not restricted to the natural representation U of the original group, that is, to the representation given by (7). Therein lies our escape route.

What we shall do is realize our group as a *blend* of copies of the natural realization (cf. [18, 19]). Let $E_1, ..., E_k$ be subspaces of the real or complex space E such that $E=E_1 \oplus \cdots \oplus E_k$. For $i=1, ..., k$ let $U_i=\langle R_{1,i}, ..., R_{n,i} \rangle$ be (not necessarily faithful) representations on E_i of some abstract group $W=\langle \sigma_1, ..., \sigma_n \rangle$, with the reflexions $R_{j,i}$ corresponding to σ_j $(j=1, ..., n)$. Extend every reflexion $R_{j,i}$ trivially to the ambient space E, so that

$U_1, ..., U_k$ act on E. Then the representation $U = \langle R_1, ..., R_n \rangle$ of W defined by

$$R_j = R_{j,1} \cdot R_{j,2} \cdot \cdots \cdot R_{j,k} \qquad (j = 1, ..., n) \tag{58}$$

is called a *blend* of $U_1, ..., U_n$. If the U_i's happen to be orthogonal representations, then we take $E_1, ..., E_k$ to be mutually orthogonal subspaces of the euclidean or unitary space E, so that U becomes also an orthogonal representation. For more details see [18].

In the situation of Sections 3–6, each representation U of the underlying abstract group $W = W(\mathscr{D})$ is associated with the matrix

$$C := (-c_{ij})_{ij}$$

defining the corresponding Hermitian form h as in (7) and (11); here, $c_{ii} := -2 (= -2a_{ii})$ for all i. Generally, different admissible choices of matrices C lead to different "natural" representations $U(C)$ (say) of W.

Let us consider one fixed natural representation, with associated matrix C. Any allowable permutation of the nodes of the corresponding diagram, which induces an outer automorphism of the group, corresponds to a permutation of the rows of C, together with the *same* permutation of its columns. This will preserve the magnitude of the c_{ij}'s, of course; in the cases we are considering, it will also preserve or conjugate products $c_{ij} c_{jk} c_{ki}$. If τ is the automorphism, denote the new matrix by C^τ; then, by an abuse of notation, $C^\tau = (-c_{\tau(i)\,\tau(j)})_{ij}$.

The group of these automorphisms is finite, since the diagram has finitely many nodes. So, blend all the corresponding representations $U(C^\tau)$ with matrices C^τ. Each automorphism τ can now be realized geometrically—just permute the coordinates within each component E_i according to τ, and then permute the components by τ.

Let us illustrate this by the example $\{\{6, 3\}_{m,0}, \{3, 6\}_{n,0}\}$ of (43) and (44). The permutations on the matrices are

$$\begin{array}{ccccc}
C = \begin{vmatrix} 2 & -c_m & -1 & -1 \\ -\bar{c}_m & 2 & -1 & -1 \\ -1 & -1 & 2 & -c_n \\ -1 & -1 & -\bar{c}_n & 2 \end{vmatrix} & \xleftrightarrow{\tau_1} & \begin{vmatrix} & -\bar{c}_m & & \\ -c_m & & & \\ & & & -c_n \\ & & -\bar{c}_n & \end{vmatrix} & = C^{\tau_1} \\
\updownarrow{\scriptstyle \tau_2} & & \updownarrow{\scriptstyle \tau_2} & \\
C^{\tau_1} = \begin{vmatrix} & -c_m & & \\ -\bar{c}_m & & & \\ & & & -\bar{c}_n \\ & & -c_n & \end{vmatrix} & \xleftrightarrow{\tau_1} & \begin{vmatrix} & -\bar{c}_m & & \\ -c_m & & & \\ & & & -\bar{c}_n \\ & & -c_n & \end{vmatrix} & = C^{\tau_1 \tau_2}.
\end{array} \tag{59}$$

Then any blend of the 4 representations corresponding to C, C^{τ_1}, C^{τ_2}, and $C^{\tau_1\tau_2}$ has the property that each τ can be realized geometrically.

Now complex conjugation has not been used above. This operation corresponds to $C \to \bar{C}(=C^T)$, and we can often employ it to halve the number of components in the blend. For example, in the above, since $C^{\tau_1\tau_2} = \bar{C}$, we can replace τ_2 (as above) by $C \to \bar{C}^{\tau_1}$, and so we need only blend the representation corresponding to C and C^{τ_1}.

Another possible simplification arises from the fact that, if we multiply the ith row (column) of C by ε_i ($\bar{\varepsilon}_i$), with $|\varepsilon_i| = 1$ ($i = 1, \ldots$), then we obtain the same geometric realization. Note that this change preserves the products $c_{ij}c_{jk}c_{ki}$.

It is worth mentioning that the blend U with components $U(C^\tau)$ can be regarded as a group of isometries with respect to a Hermitian form g, which in a sense is the blend of the Hermitian forms defined by the C^τ. The form g is defined by a matrix with blocks C^τ arranged along the diagonal and with all other entries equal to 0.

In the above construction each group $U(C^\tau)$ is abstractly isomorphic to $U(C)$, and so is the blend U. In particular this implies that for each outer automorphism τ of the underlying group W there is a corresponding outer automorphism of the (abstract) group $U(C)$.

> It follows that in Sections 3 to 6 each group U (which is related to a diagram) can itself be used for the construction of a regular incidence-polytope. (60)

The polytope is obtained from the same twisting operation as for the corresponding abstract group W; its group is the semi-direct product of U by the corresponding group of outer automorphisms.

Notwithstanding our above remarks, sometimes polytopes admit realizations which suggest that the full group does occur as a group of isometries with respect to h, even though it is not immediately obvious how.

For example, consider the group $U = \langle R_1, \ldots, R_4 \rangle$ constructed in Section 6 in connexion with the universal $\mathscr{R}_{m,3m} = \{\{3,6\}_{m,m}, \{3,6\}_{3m,0}\}$, $m \geqslant 2$. As we saw above, there exist outer automorphisms τ_1, τ_2 (say) of U such that the operation

$$\kappa: (R_1, \ldots, R_4; \tau_1, \tau_2) \to (\tau_1, \tau_2, R_3, R_4) =: (\sigma_0, \ldots, \sigma_3)$$

gives the group $\langle \sigma_0, \ldots, \sigma_3 \rangle$ of a regular 4-incidence-polytope $\mathscr{P}$ in the class $\langle \{3,6\}_{m,m}, \{3,6\}_{3m,0} \rangle$. Apply Wythoff's construction combinatorially to the dual $\mathscr{P}^*$ of $\mathscr{P}$, with initial vertex the vertex F_0 in the base flag of $\mathscr{P}^*$. This gives combinatorial information about the incidence of faces in $\mathscr{P}^*$. In particular, note that $A(\mathscr{P}^*) = U \cdot \langle \tau_1, \tau_2 \rangle$ and that τ_1 and τ_2 fix F_0 as well as the edge F_1 in the base flag of $\mathscr{P}^*$; it follows that the transforms of F_0 and F_1 by U cover all vertices and edges of $\mathscr{P}^*$, respectively.

Now, if Wythoff's construction is applied geometrically to U by ringing node 4 of the corresponding diagram (53) (that is, by picking a point in $(R_1 \cap R_2 \cap R_3)\backslash R_4$), then clearly all the transforms of F_0 and F_1 by U give the vertices and edges of a geometrical realization of $\mathscr{P}^*$. However, the (geometrical) transforms by U of the 2-face and 3-face in the base flag will not cover all 2-faces and 3-faces, respectively; instead, one has to employ here the additional information on the structure of $\mathscr{P}^*$ obtained from the combinatorial version of Wythoff's construction. This is an instance of a polytope, where we can find a realization even if we may not be sure how to realize geometrically the outer automorphisms.

For the polytopes related to $\{\{6, 3\}_{m,0}, \{3, 6\}_{n,0}\}$ (obtained from (43)), the situation is less clear. Here the vertices of the polytope $\mathscr{P}$ constructed from U are those obtained by ringing node 1 in diagram (43) together with those obtained by ringing node 2 $(=\tau_1(1))$, so that only a realization of τ_2 need be found—the geometry will take care of τ_1. But, of course, the realization of τ_2 alone was the problem.

8. The Case $\{6, 3\}_{1,1}$

In Sections 3 to 7 the maps $\{6, 3\}_{1,1}$ and $\{3, 6\}_{1,1}$ had been excluded from the discussion because of the lack of appropriate Hermitian forms. In this section we complete the classification by dealing with these two cases.

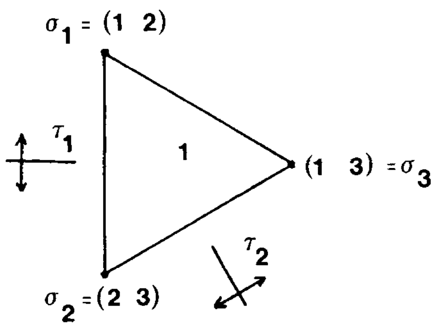
We begin by observing that $\{6, 3\}_{1,1}$ can be constructed as in (24) and (25), with $m = 1$. Now for $m = 1$ the relation (4) becomes $\sigma_1\sigma_2\sigma_3\sigma_2 = 1$, or equivalently,

$$\sigma_1 = \sigma_2\sigma_3\sigma_2;$$

at the same time,

$$\sigma_2 = \sigma_1\sigma_3\sigma_1, \qquad \sigma_3 = \sigma_1\sigma_2\sigma_1.$$

This implies that $W := \langle \sigma_1, \sigma_2, \sigma_3 \rangle = S_3$, as illustrated in



(61)

In particular, the group of $\{6, 3\}_{1,1}$ is a semi-direct product of S_3 by S_3.

Now, let $\mathcal{M}$ be a regular map of type $\{3, p\}$, which is an incidence-polytope. Assume the universal $\mathcal{P} = \{\{6, 3\}_{1,1}, \mathcal{M}\}$ exists. We shall show that then the (vertex set of the) edge graph of $\mathcal{M}$ must be 3-colourable.

Let $A(\mathcal{P}) = \langle \rho_0, ..., \rho_3 \rangle$, with ρ_1, ρ_2, ρ_3 the distinguished generators of a base flag $\{G_0, G_1, G_2\}$ (say) of $\mathcal{M}$. It is easy to see that

$$A(\mathcal{P}) = N_0 \cdot \langle \rho_1, \rho_2, \rho_3 \rangle, \tag{62}$$

with N_0 the normal closure of ρ_0 in $A(\mathcal{P})$; in fact, N_0 is generated by the conjugates $\varphi \rho_0 \varphi^{-1}$ with $\varphi \in \langle \rho_1, \rho_2, \rho_3 \rangle$ (cf. [26, p. 308]). In a sense, N_0 plays the role of W in Section 5.2. In particular, we shall see that $N_0 = S_3$.

Let $V(\mathcal{M})$ denote the vertex set of $\mathcal{M}$. We shall label the vertices of $\mathcal{M}$ by conjugates $\varphi \rho_0 \varphi^{-1}$ with $\varphi \in \langle \rho_1, \rho_2, \rho_3 \rangle$. Define the labelling λ by

$$\begin{cases} \lambda \colon V(\mathcal{M}) \to N_0 \\ \quad \varphi(G_0) \to \varphi \rho_0 \varphi^{-1} \qquad (\varphi \in \langle \rho_1, \rho_2, \rho_3 \rangle). \end{cases} \tag{63}$$

The map λ is well-defined; in fact, if $\varphi(G_0) = G_0$, then $\varphi \in \langle \rho_2, \rho_3 \rangle$, so that $\varphi \rho_0 \varphi^{-1} = \rho_0$ in $A(\mathcal{P})$. Furthermore, λ commutes with the action of $\langle \rho_1, \rho_2, \rho_3 \rangle$; that is, if $\psi \in \langle \rho_1, \rho_2, \rho_3 \rangle$, then

$$\lambda(\psi(\varphi(F_0))) = \psi \cdot \lambda(\varphi(F_0)) \cdot \psi^{-1}$$

for all $\varphi \in \langle \rho_1, \rho_2, \rho_3 \rangle$.

Now, let $\{F, G, H\}$ and $\{G, H, K\}$ be the vertex sets of two adjacent 2-faces of $\mathcal{M}$. Then there exists an involution $\psi \in \langle \rho_1, \rho_2, \rho_3 \rangle$ such that $\psi(G) = G$, $\psi(H) = H$, and $\psi(F) = K$. By the properties of λ,

$$\psi \cdot \lambda(G) \cdot \psi^{-1} = \lambda(G), \qquad \psi \cdot \lambda(H) \cdot \psi^{-1} = \lambda(H), \text{ and } \psi \cdot \lambda(F) \cdot \psi^{-1} = \lambda(K);$$

that is, the labels of $\{F, G, H\}$ are converted to those of $\{G, H, K\}$ by conjugation with ψ, and this conjugation leaves two of the labels invariant.

If $\{F, G, H\} = \{G_0, G_1, G_2\}$, then

$$\lambda(G_0) = \rho_0 (= \sigma_1), \qquad \lambda(G_1) = \rho_1 \rho_0 \rho_1 (= \sigma_2),$$

and

$$\lambda(G_2) = \rho_2 \rho_1 \rho_0 \rho_1 \rho_2 (= \sigma_3);$$

they generate a group S_3 as in (61). But as pointed out above, in S_3 any generator σ_i is determined by the two others; hence, if two generators remained fixed under conjugation with ψ, then so does the third. Thus, the set of labels on any adjacent 2-face of $\{G_0, G_1, G_2\}$ is again ρ_0, $\rho_1 \rho_0 \rho_1$, and $\rho_2 \rho_1 \rho_0 \rho_1 \rho_2$. Hence, by the connectivity of $\mathcal{M}$, this is true for all 2-faces of $\mathcal{M}$. It follows that the edge graph of $\mathcal{M}$ is 3-colourable.

As an immediate consequence, the universal $\mathscr{P} = \{\{6, 3\}_{1,1}, \mathscr{M}\}$ cannot exist for $\mathscr{M} = \{3, 3\}$, $\{3, 5\}$, and $\{3, 6\}_{m,0}$ with $3 \nmid m$. We proceed by showing that $\mathscr{P}$ exists if $\mathscr{M} = \{3, 4\}$, $\{3, 6\}_{3m,0}$ with $m \geqslant 1$, or $\{3, 6\}_{m,m}$ with $m \geqslant 1$.

More generally, let the edge graph of $\mathscr{M}$ be 3-colourable. As seen above, if $\mathscr{P} = \{\{6, 3\}_{1,1}, \mathscr{M}\}$ exists, then the labels $\varphi\rho_0\varphi^{-1}$ of the vertices are $\rho_0, \rho_1\rho_0\rho_1$ or $\rho_2\rho_1\rho_0\rho_1\rho_2$; hence $N_0 = \langle \rho_0, \rho_1\rho_0\rho_1, \rho_2\rho_1\rho_0\rho_1\rho_2 \rangle = S_3$. Since $N_0 \subset \langle \rho_1, \rho_2, \rho_3 \rangle$, the product in (62) must be semi-direct, so that $A(\mathscr{P})$ must be a semi-direct product of S_3 by $A(\mathscr{M})$.

In order to construct $\mathscr{P}$, 3-colour the vertices of $\mathscr{M}$. Then consider a diagram $\mathscr{D}$ whose nodes are the vertices of $\mathscr{M}$; connect any two nodes by an unlabelled branch if and only if they have different colours; finally, span each triple of distinctly coloured vertices of $\mathscr{M}$ by a triangular 2-face and mark it by 1. (That is, in general the underlying simplicial 2-complex for $\mathscr{D}$ is obtained from $\mathscr{M}$ by adding further edges and 2-faces.) Similar considerations to those above show that the abstract group $W = W(\mathscr{D}) = \langle \sigma_F \mid F \text{ a vertex of } \mathscr{M} \rangle$ is isomorphic to S_3; the isomorphism can be established by "colouring" the vertices of $\mathscr{M}$ by the transpositions (1 2), (2 3), and (1 3). By construction $A(\mathscr{M})$ acts on W as a group of automorphisms permuting the generators σ_F of W.

Now, pick a base flag $\{F_0, F_1, F_2\}$ for $\mathscr{M}$; let τ_0, τ_1, τ_2 be the corresponding distinguished generators of $A(\mathscr{M})$. Construct the group $A = \langle \rho_0, ..., \rho_3 \rangle$ as in Section 5.2 by the operation

$$\kappa\colon (\{\sigma_F\}_F; \tau_0, \tau_1, \tau_2) \to (\sigma_{F_0}, \tau_0, \tau_1, \tau_2) = (\rho_0, ..., \rho_3).$$

As in Section 5.2 it can be shown that A is a C-group. By construction, A is a semi-direct product of W $(= S_3)$ by $A(\mathscr{M})$. It follows from our above consideration that the regular 4-incidence-polytope with group A must be the universal $\{\{6, 3\}_{1,1}, \mathscr{M}\}$. We summarize the results in a theorem.

THEOREM 7. *Let $\mathscr{M}$ be a regular map of type $\{3, p\}$.*

(a) *The universal $\{\{6, 3\}_{1,1}, \mathscr{M}\}$ exists if and only if the edge graph of $\mathscr{M}$ is 3-colourable.*

(b) *Let the edge graph of $\mathscr{M}$ be 3-colourable. The group of $\{\{6, 3\}_{1,1}, \mathscr{M}\}$ is a semi-direct product of S_3 by $A(\mathscr{M})$, and so is finite if and only if $\mathscr{M}$ is finite.*

COROLLARY. *The universal $\{\{6, 3\}_{1,1}, \mathscr{M}\}$*

(a) *exists for $\mathscr{M} = \{3, 4\}$, $\{3, 6\}_{m,m}$ with $m \geqslant 1$, and $\{3, 6\}_{3m,0}$ with $m \geqslant 1$;*

(b) *does not exist for $\mathscr{M} = \{3, 3\}$, $\{3, 5\}$, and $\{3, 6\}_{m,0}$ with $3 \nmid m$.*

The above proof of Theorem 7 and its corollary is inspired by the methods of [26], especially the constructions involving the so-called degenerate amalgamation property (DAP) of a regular incidence-polytope. A regular d-incidence-polytope $\mathscr{L}$ with group $A(\mathscr{L})=\langle \rho_0, ..., \rho_{d-1}\rangle$ is said to have the *DAP with respect to its vertex-figures* if $A(\mathscr{L})$ is a semi-direct product of the normal closure N_0 of ρ_0 in $A(\mathscr{L})$ by the subgroup $\langle \rho_1, ..., \rho_{d-1}\rangle$. Similarly, $\mathscr{L}$ has the *DAP with respect to its facets* if $\mathscr{L}^*$ has the DAP with respect to its vertex-figures.

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two regular d-incidence-polytopes such that the vertex-figures of $\mathscr{L}_1$ are isomorphic to the facets of $\mathscr{L}_2$. If $\mathscr{L}_1$ has the DAP with respect to its vertex-figures and $\mathscr{L}_2$ with respect to its facets, then there exists a regular $(d+1)$-incidence-polytope $\mathscr{L}$ of the class $\langle \mathscr{L}_1, \mathscr{L}_2\rangle$ which is *combinatorially flat*; that is, each vertex of $\mathscr{L}$ is incident with each facet of $\mathscr{L}$ (cf. [26, Theorem 2]). In particular, in this situation, the universal $\{\mathscr{L}_1, \mathscr{L}_2\}$ exists. It was shown in [26] that the regular maps $\mathscr{M}=\{3, 4\}$, $\{3, 6\}_{m,m}$ and $\{3, 6\}_{3m,0}$ all have the DAP with respect to their facets. Thus for these maps the existence of the universal $\{\{6, 3\}_{1,1}, \mathscr{M}\}$ follows from the results of [26]. Furthermore, comparing the groups shows that the combinatorially flat 4-incidence-polytope $\mathscr{L}$ of the class $\langle\{6, 3\}_{1,1}, \mathscr{M}\rangle$ is indeed the universal $\{\{6, 3\}_{1,1}, \mathscr{M}\}$. This fact holds more generally for all regular maps of type $\{3, p\}$ for which the edge-graph is 3-colourable. In fact, it can be shown that the 3-colourability is equivalent to the DAP.

Concluding, consider the case of the universal $\mathscr{R}_{1,n}=\{\{3, 6\}_{1,1}, \{6, 3\}_{n,0}\}$ for $n\geqslant 2$. As in Section 6, if $\mathscr{R}_{1,n}$ exists, then its group is obtained from the group W with diagram (53) (with $m=1$) by operation (54). But the mark $m=1$ forces $\sigma_3=\sigma_1\sigma_2\sigma_1$, so that $W=\langle\sigma_1, \sigma_2, \sigma_4\rangle=[1\ \ 1\ \ 1]^n$. In particular,

$$\sigma_4\sigma_3\sigma_2\sigma_3=\sigma_4\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1=\sigma_4\sigma_1,$$

so that necessarily $n=3$. Conversely, if $n=3$, then starting from $[1\ \ 1\ \ 1]^n=\langle\sigma_1, \sigma_2, \sigma_4\rangle$ and introducing a new generator $\sigma_3=\sigma_1\sigma_2\sigma_1$ gives indeed the group W with diagram (53) (with $m=1$). Hence, we find that $\mathscr{R}_{1,n}$ exists if and only if $n=3$; the group of $\mathscr{R}_{1,3}$ is a semi-direct product of $[1\ \ 1\ \ 1]^3$ by S_3, of order 324.

We then observe that $\{\{3, 6\}_{1,1}, \{6, 3\}_{1,1}\}$ also exists, and is finite, since it can be obtained from $\{\{3, 6\}_{1,1}, \{6, 3\}_{3,0}\}$ by identifications. The universal $\{\{3, 6\}_{1,1}, \{6, 3\}_{1,1}\}$ is combinatorially flat and has a group of order 108.

Remark. In the forthcoming paper [23] we shall discuss various relationships between the universal polytopes of type $\{6, 3, 3\}$, $\{3, 6, 3\}$, and $\{6, 3, 6\}$. Also, we shall give further geometrical explanations for the finiteness and non-finiteness of the polytopes.

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